Multiplicity of positive solutions for two coupled nonlinear Schrödinger equations

Tsung-fang Wu
Department of Applied Mathematics
National University of Kaohsiung, Kaohsiung, Taiwan

(joint work with Tai-Chia Lin)
Outline

• The History of BEC
• Motivation
• Nehari manifold
• Palais-Smale sequences
• Our main result
1 The History of BEC
2 Motivation

We consider a two-component system of nonlinear Schrödinger equations with optical lattice potentials:

\[
\begin{aligned}
\frac{\hbar^2}{2m} \triangle \Phi_j - \tilde{V}_j(x) \Phi_j + \sum_{i=1}^2 \tilde{\beta}_{ij} |\Phi_i|^2 \Phi_j = - i \hbar \partial_t \Phi_j & \quad \text{for } x \in \Omega, \\
\Phi_j(x, t) = 0 & \quad \text{for } x \in \partial \Omega, \\
t > 0, \ j = 1, 2,
\end{aligned}
\]

(2.1)

where \( \Omega \subseteq \mathbb{R}^N, \ N = 2, 3 \) is the region for condensate dwelling, \( \Phi_j \)'s are corresponding condensate wave functions, \( \hbar \) is Planck constant, \( m \) is the atomic mass, \( \tilde{\beta}_{jj} = -(N_j - 1)U_{jj} > 0 \) and \( \tilde{\beta}_{12} = \tilde{\beta}_{21} = -N_2U_{12} \).
Here each $N_j \geq 1$ is a fixed number of atoms in the hyperfine state $|j\rangle$, and $U_{ij} = 4\pi \frac{\hbar^2}{2m} a_{ij}$, where $a_{jj}$‘s and $a_{12}$ are the intraspecies and interspecies scattering lengths. $\tilde{V}_j$ is the optical lattice potential for the $j$-th species and is a periodic function of spatial variables written as (see [Jaksch etc., Phys. Rev. Lett., 1998]):

$$\tilde{V}_j(x) = \sum_{k=1}^{N} \tilde{\nu}_{j,k} \sin^2(l x_k) \quad \text{for} \quad x = (x_1, \cdots, x_N) \in \Omega, j = 1, 2,$$

(2.2)

where $\tilde{\nu}_{j,k} \geq 0$ is the associated potential depth, $l = 2\pi/L$, and $L$ is the wavelength of the laser light.
Here, we study steady state bright solitons with the form
\( \Phi_j = e^{i\tilde{\lambda}_j \frac{t}{\hbar}} u_j(x) \), \( j = 1, 2 \) called bound states of the system (2.1),
where \( \tilde{\lambda}_j \)'s are positive constants and \( u_j \)'s satisfy
\[
\begin{aligned}
\frac{\hbar^2}{2m} \Delta u_j(x) - \tilde{V}_j(x) u_j(x) + \sum_{i=1}^{2} \tilde{\beta}_{ij} u_i^2(x) u_j(x) &= \tilde{\lambda}_j u_j \quad \text{in } \Omega, \\
u_j &> 0 \quad \text{in } \Omega, \\
u_j &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]
(2.3)
Physically, \( u_j \)'s are the associated condensate amplitudes of the bound states \( \Phi_j \)'s, and \( \tilde{\lambda}_j \)'s are chemical potentials. Due to Feshbach resonance (see [Inouye etc., Nature, 1998]), we may set \( \tilde{\beta}_{jj} \)'s, \( \tilde{\lambda}_j \)'s and \( \tilde{\nu}_{j,k} \)'s to be very large quantities.
By rescaling and some simple assumptions, the problem (2.3) with very large $\tilde{\beta}_{jj}$’s, $\tilde{\lambda}_j$’s and $\tilde{\nu}_{j,k}$’s is equivalent to the following singularly perturbed problem:

$$
\begin{cases}
\epsilon^2 \Delta u_j - \hat{V}_j(x)u_j + \sum_{i=1}^{2} \beta_{ij} u_i^2 u_j = 0 & \text{in } \Omega, \\
u_j > 0 & \text{in } \Omega, \\
u_j = 0 & \text{on } \partial \Omega, \; j = 1, 2,
\end{cases}
$$

(2.4)

where $0 < \epsilon \ll 1$ is a small parameter, and $\hat{V}_j$’s are defined by

$$
\hat{V}_j(x) = \lambda_j + \sum_{k=1}^{N} \nu_{j,k} \sin^2(l x_k), \quad \text{for } x = (x_1, \cdots, x_N) \in \Omega.
$$

(2.5)

Here $\lambda_j > 0$, $\nu_{j,k} \geq 0$, $l > 0$, and $\beta_{ij}$’s are constants independent of $\epsilon$. 

5
Problems

To see how the potentials act on the existence and number of steady state bright solitons (i.e. ground states or bound states). i.e. we study the existence and multiplicity of positive solutions of problem (2.4).
Without loss of generality, problem (2.4) can be generalized to the following problem:

\[
\begin{align*}
\varepsilon^2 \Delta u - V_1(x) u + \mu_1 u^3 + \beta u v^2 &= 0 & \text{in } \mathbb{R}^N, \\
\varepsilon^2 \Delta v - V_2(x) v + \mu_2 v^3 + \beta u^2 v &= 0 & \text{in } \mathbb{R}^N, \\
u > 0, v > 0 & & \text{in } \mathbb{R}^N,
\end{align*}
\]  

(Eε)

where \( N = 1, 2, 3 \), and the potentials \( V_1, V_2 \in C(\mathbb{R}^N) \) satisfy

(I) \( V_1, V_2 \) are bounded functions on \( \mathbb{R}^N \) and
\[
\Lambda_l = \sup_{x \in \mathbb{R}^N} |V_l(x)| < \infty, l = 1, 2.
\]

(II) The global minima of potentials \( V_l, l = 1, 2 \) only occur at \( \kappa \)
points \( a_1, a_2, \ldots, a_\kappa \in \mathbb{R}^N \) if \( l = 1 \), and \( m \) points \( b_1, b_2, \ldots, b_m \in \mathbb{R}^N \) if \( l = 2 \), respectively.
Moreover,

\[ V_1 (a_i^i) = \lambda_1 = \min \left\{ V_1 (x) : x \in \mathbb{R}^N \right\} > 0, \quad i = 1, 2, \ldots, \kappa, \]

and

\[ V_2 (b_j^j) = \lambda_2 = \min \left\{ V_2 (x) : x \in \mathbb{R}^N \right\} > 0, \quad j = 1, 2, \ldots, m. \]
3 Nehari manifold

Let \((u,v) \in H\) be a solution of problem \((E_\epsilon)\). Then

\[
\int_{\mathbb{R}^N} \epsilon^2 \nabla u \nabla \phi_1 + \int_{\mathbb{R}^N} V_1 u \phi_1 + \int_{\mathbb{R}^N} \epsilon^2 \nabla v \nabla \phi_2 + \int_{\mathbb{R}^N} V_2 v \phi_2 - \int_{\mathbb{R}^N} \mu_1 u^3 \phi_1 \\
- \int_{\mathbb{R}^N} \mu_2 v^3 \phi_2 - \beta \int_{\mathbb{R}^N} uv^2 \phi_1 - \beta \int_{\mathbb{R}^N} u^2 v \phi_2 = 0, \quad \forall (\phi_1, \phi_2) \in H,
\]

where \(H = H^1 (\mathbb{R}^N) \times H^1 (\mathbb{R}^N)\) is a Sobolev space with the norm \(\| \cdot \|_H\) given by

\[
\|(\phi, \psi)\|^2_H = \|\phi\|_{V_1}^2 + \|\psi\|_{V_2}^2 \\
= \int_{\mathbb{R}^N} \left( \epsilon^2 |\nabla \phi|^2 + V_1 \phi^2 \right) + \int_{\mathbb{R}^N} \left( \epsilon^2 |\nabla \psi|^2 + V_2 \psi^2 \right).
\]
It is well known that the solution \((u,v) \in H\) of problem \((E_\varepsilon)\) is a critical point of the energy functional \(J_\varepsilon \in C^1 (H, \mathbb{R})\) defined by

\[
J_\varepsilon (\phi, \psi) = \frac{1}{2} \|(\phi, \psi)\|^2_H - \frac{1}{4} \left( \int_{\mathbb{R}^N} \mu_1 \phi^4 + \int_{\mathbb{R}^N} \mu_2 \psi^4 \right) - \frac{\beta}{2} \int_{\mathbb{R}^N} \phi^2 \psi^2
\]

for all \((\phi, \psi) \in H\).

Note that the energy functional \(J_\varepsilon\) is not bounded below on \(H\), it is useful to consider the functional on the Nehari manifold.
A. codimension one

\[ \tilde{N}_\varepsilon = \{ (u, v) \in H \setminus \{(0, 0)\} : \langle J'_{\varepsilon}(u, v), (u, v) \rangle = 0 \} \]

where

\[ \langle J'_{\varepsilon}(u, v), (u, v) \rangle = \|(u, v)\|_H^2 - \left( \mu_1 \int_{\mathbb{R}^N} u^4 + \mu_2 \int_{\mathbb{R}^N} v^4 \right) - 2\beta \int_{\mathbb{R}^N} u^2 v^2. \]

- trivial solution \((0, 0) \notin \tilde{N}_\varepsilon\)
- semitrivial solutions \((u, 0), (0, v) \in \tilde{N}_\varepsilon\)
- nontrivial solution \((u, v) \in \tilde{N}_\varepsilon\)
B. codimension two

\[\mathbf{N}_\varepsilon = \left\{ (u, v) \in \mathbf{T} : \quad \begin{align*}
\|u\|^2_{V_1} &= \mu_1 \int_{\mathbb{R}^N} u^4 + \beta \int_{\mathbb{R}^N} u^2 v^2 \\
\|v\|^2_{V_2} &= \mu_2 \int_{\mathbb{R}^N} v^4 + \beta \int_{\mathbb{R}^N} u^2 v^2
\end{align*} \right\},\]

where \(\mathbf{T} = \{(u, v) \in H : u \neq 0 \text{ and } v \neq 0\}\).

- trivial solution \((0, 0) \notin \mathbf{N}_\varepsilon\)
- nontrivial solution \((u, v) \in \mathbf{N}_\varepsilon\)
**Example:** Let $V_l \equiv \lambda_l = 1 = \mu_l$ for $l = 1, 2$ and $\beta \in \mathbb{R}$. Define $\alpha_{\lambda_l, \mu_l}$ is the minimizing energy of the functional $I_{\lambda_l, \mu_l}$ over the Nehari manifold $M_{\lambda_l, \mu_l}$ defined by

\[
\alpha_{\lambda_l, \mu_l} = \inf \{ I_{\lambda_l, \mu_l} (u) \mid u \in M_{\lambda_l, \mu_l} \},
\]
\[
I_{\lambda_l, \mu_l} (u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda_l |u|^2) - \frac{\mu_l}{4} \int_{\mathbb{R}^N} |u|^4, \quad \forall : u \in H^1 (\mathbb{R}^N)
\]
\[
M_{\lambda_l, \mu_l} = \{ u \in H^1 (\mathbb{R}^N) \setminus \{0\} \mid \langle I'_{\lambda_l, \mu_l} (u), u \rangle = 0 \},
\]
for $l = 1, 2$. 

13
Thus we should choose the Nehari manifold with codimension two.

**Lemma 1.** Let \((u_0, v_0)\) be a constrained critical point of \(J_\varepsilon\) on \(N_\varepsilon\) with

\[
\mu_1 \mu_2 \int_{\mathbb{R}^N} u_0^4 \int_{\mathbb{R}^N} v_0^4 - \beta^2 \left( \int_{\mathbb{R}^N} u_0^2 v_0^2 \right)^2 > 0.
\]

Then \(\nabla J_\varepsilon (u_0, v_0) = 0\) on \(H^*\).

**proof:** Define

\[
\begin{align*}
f(u, v) &= \|u\|_{V_1}^2 - \mu_1 \int_{\mathbb{R}^N} u^4 - \beta \int_{\mathbb{R}^N} u^2 v^2, \\
g(u, v) &= \|v\|_{V_2}^2 - \mu_2 \int_{\mathbb{R}^N} v^4 - \beta \int_{\mathbb{R}^N} u^2 v^2.
\end{align*}
\]

Since \((u_0, v_0)\) is a constrained critical point of \(J_\varepsilon\) on \(N_\varepsilon\), by the Lagrange multiplier, there exist \(\theta_1, \theta_2 \in \mathbb{R}\) such that

\[
\nabla J_\varepsilon (u_0, v_0) = \theta_1 \nabla f (u_0, v_0) + \theta_2 \nabla g (u_0, v_0).
\]
Then we have
\[\theta_1 \left[ \|u_0\|_{V^1}^2 - 2\mu_1 \int_{\mathbb{R}^N} u_0^4 - 2\beta \int_{\mathbb{R}^N} u_0^2 v_0^2 \right] - \theta_2 \left[ \beta \int_{\mathbb{R}^N} u_0^2 v_0^2 \right] = 0,\]
\[-\theta_1 \left[ \beta \int_{\mathbb{R}^N} u_0^2 v_0^2 \right] + \theta_2 \left[ \|v_0\|_{V^2}^2 - 2\mu_2 \int_{\mathbb{R}^N} v_0^4 - 2\beta \int_{\mathbb{R}^N} u_0^2 v_0^2 \right] = 0\]
or
\[\Sigma \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\]
where
\[\Sigma = \begin{bmatrix} \mu_1 \int_{\mathbb{R}^N} u_0^4 & \beta \int_{\mathbb{R}^N} u_0^2 v_0^2 \\ \beta \int_{\mathbb{R}^N} u_0^2 v_0^2 & \mu_2 \int_{\mathbb{R}^N} v_0^4 \end{bmatrix}.\]
Then
\[\det \Sigma = \mu_1 \mu_2 \int_{\mathbb{R}^N} u_0^4 \int_{\mathbb{R}^N} v_0^4 - \beta^2 \left( \int_{\mathbb{R}^N} u_0^2 v_0^2 \right)^2 > 0,\]
this implies \(\theta_1 = \theta_2 = 0\). Therefore \(\nabla J_\epsilon(u_0, v_0) = 0\) on \(H^*\).
Now we state results which are useful to prove our main result as follows:

**Lemma 2.** Let $\beta < 0$, $\varepsilon, \sigma > 0$ and $(u, v) \in N_\varepsilon$. If

$$\int_{\mathbb{R}^N} u^2 v^2 \geq \sigma \varepsilon^{N},$$

then

$$\varepsilon^{-N} J_\varepsilon (u, v) > \alpha_{\lambda_1, \mu_1} + \alpha_{\lambda_2, \mu_2} + \bar{\delta}$$

where

$$\bar{\delta} = \frac{|\beta| \sigma}{2} \min \left\{ 1, \frac{1}{2} \min \left\{ \frac{S_1^2}{\mu_1 (4\alpha_{\lambda_1, \mu_1} + |\beta| \sigma)}, \frac{S_2^2}{\mu_2 (4\alpha_{\lambda_2, \mu_2} + |\beta| \sigma)} \right\} \right\}$$

and

$$S_i = \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \nabla u^2 + \lambda_i u^2}{(\int_{\mathbb{R}^N} u^4)^{1/2}} > 0, \quad i = 1, 2.$$
Furthermore, if \( 0 < \sigma < \frac{4\sqrt{\alpha_1,\mu_1,\alpha_2,\mu_2}}{|\beta|} \), then

\[
\left[ \mu_1 \mu_2 \int_{\mathbb{R}^N} u^4 \int_{\mathbb{R}^N} v^4 - \beta^2 \left( \int_{\mathbb{R}^N} u^2 v^2 \right)^2 \right] > \varepsilon^{2N} \left( 16\alpha_{\lambda_1,\mu_1} \alpha_{\lambda_2,\mu_2} - \beta^2 \sigma^2 \right) > 0
\]

for all \((u, v) \in \mathbb{N}_\varepsilon\) and \(\varepsilon^{-N} J_\varepsilon (u, v) \leq \alpha_{\lambda_1,\mu_1} + \alpha_{\lambda_2,\mu_2} + \delta\).
When the potentials $V_l$’s satisfy
\[
0 < \lambda_l = \inf_{x \in \mathbb{R}^N} V_l(x) < \lim_{|x| \to \infty} V_l(x) \leq \infty \quad \text{for } l = 1, 2, \quad (3.5)
\]
Lin and Wei study the minimization of the functional $J_\epsilon$ over the manifold $N_\epsilon$. As $\beta < 0$ and $\epsilon > 0$ sufficiently small, there exists a least energy solution $(u_{\epsilon,1}, u_{\epsilon,2})$ of problem $(E_\epsilon)$ such that
\[
\epsilon^{-N} J_\epsilon(u_{\epsilon,1}, u_{\epsilon,2}) \to \alpha_{\lambda_1, \mu_1} + \alpha_{\lambda_2, \mu_2}, \quad (3.6)
\]
as $\epsilon$ goes to zero (up to a subsequence).
Remark: The our conditions (I) and (II) may allow the case that

\[
\liminf_{|x| \to \infty} V_l(x) < \limsup_{|x| \to \infty} V_l(x) \quad \text{for some } l = 1, 2.
\]

Hence the argument of Lin-Wei may not be applicable to problem \((E_{\varepsilon})\) with the conditions (I) and (II).
4 Palais-Smale sequences

To find Palais-Smale (PS) sequences of the functional $J_\varepsilon$, we may use the conditions (I) and (II) of potentials $V_\varepsilon$’s and a generalized barycenter map given by $\Phi : L^2(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ a continuous map satisfying

$$\Phi(\xi * u) = \xi + \Phi(u) \quad \text{and} \quad \Phi(u \circ A^{-1}) = A\Phi(u),$$

(4.1)

and

$$\Phi(u \circ \varepsilon) = \varepsilon^{-1}\Phi(u) \quad \text{for all} \ u \in L^2(\mathbb{R}^N) \setminus \{0\} \quad \text{and} \ \varepsilon > 0.$$  

(4.2)

for every $\xi \in \mathbb{R}^N$, every orthogonal $N \times N$ matrix $A$, every $\varepsilon > 0$ and every $u \in L^2(\mathbb{R}^N) \setminus \{0\}$, where $(\xi * u)(x) = u(x - \xi)$ and $(u \circ \varepsilon)(x) = u(\varepsilon x)$. 
We may use the map $\Phi$ to decompose the Nehari manifold $N_\varepsilon$ into $\kappa \times m$ submanifolds $N_{i,j} (\varepsilon)$’s as follows:

$$N_{i,j} (\varepsilon) = \left\{ (u, v) \in N_\varepsilon : \Phi (u) \in C_l (a^i) \text{ and } \Phi (v) \in C_l (b^j) \right\},$$

with the boundary

$$O_{i,j} (\varepsilon) = \left\{ (u, v) \in N_\varepsilon : \Phi (u) \in \partial C_l (a^i) \text{ or } \Phi (v) \in \partial C_l (b^j) \right\},$$

for $i = 1, 2, \ldots, \kappa$ and $j = 1, 2, \ldots, m$. Hereafter, $C_l (x)$’s are cubes defined by $C_l (x) = \prod_{i=1}^{N} (x_i - l, x_i + l)$ with the boundary $\partial C_l (x)$ for $l > 0$ and $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$. 
By the conditions (I) and (II), there exists \( l > 0 \) such that \( \{C_l(a^i)\}_{i=1}^{\kappa} \) and \( \{C_l(b^j)\}_{j=1}^{m} \) are collections of disjoint cubes satisfying

\[
\begin{align*}
V_1(x) &> V_1(a^i) & \text{for } x \in \partial C_l(a^i) \text{ and } i = 1, 2, \ldots, \kappa, \\
V_2(x) &> V_2(b^j) & \text{for } x \in \partial C_l(b^j) \text{ and } j = 1, 2, \ldots, m, \\
C_l(a^i) \cap C_l(b^j) &\neq \emptyset & \text{if } a^i \neq b^j, \\
C_l(a^i) &= C_l(b^j) & \text{if } a^i = b^j.
\end{align*}
\]

Now we consider the minimization of the functional \( J_\varepsilon \) over \( N_{i,j}(\varepsilon) \) and \( O_{i,j}(\varepsilon) \), respectively, and denote the corresponding minima as

\[
\gamma_{i,j}(\varepsilon) = \inf_{(u,v) \in N_{i,j}(\varepsilon)} J_\varepsilon(u,v) \quad \text{and} \quad \tilde{\gamma}_{i,j}(\varepsilon) = \inf_{(u,v) \in O_{i,j}(\varepsilon)} J_\varepsilon(u,v).
\]

(4.3)
Existence of Palais-Smale sequences

**Proposition.** For each $i \in \{1,2,\ldots,\kappa\}, j \in \{1,2,\ldots,m\}$, there exists a sequence of functions $\{(u_n,v_n)\}_{n=1}^{\infty} \subset N_{i,j}(\varepsilon)$ such that $u_n, v_n \geq 0$ in $\mathbb{R}^N$ for $n \in \mathbb{N}$,

$$J_\varepsilon(u_{i,j}^n,v_{i,j}^n) \to \gamma_{i,j}(\varepsilon) \text{ in } H \text{ as } n \to \infty$$

and

$$J'_\varepsilon(u_{i,j}^n,v_{i,j}^n) \to 0 \text{ in } H^* \text{ as } n \to \infty.$$
Proposition. Let $i \in \{1, \ldots, \kappa\}$ and $j \in \{1, \ldots, m\}$. Assume
\[
\{(u_{i,j}^n, v_{i,j}^n)\}_{n=1}^\infty \text{ is a sequence in } N_{i,j}(\varepsilon) \text{ satisfying } u_{i,j}^n, v_{i,j}^n \geq 0 \text{ in } \mathbb{R}^N \text{ for } n \in \mathbb{N},
\]
(i) $J_\varepsilon(u_{i,j}^n, v_{i,j}^n) \to \gamma_{i,j}(\varepsilon)$ as $n \to \infty$;
(ii) $J'_\varepsilon(u_{i,j}^n, v_{i,j}^n) \to 0$ strongly in $H^*$ as $n \to \infty$.

Then there exists a convergent subsequence denoted also as
\[
\{(u_{i,j}^n, v_{i,j}^n)\}_{n=1}^\infty \text{ for notation convenience such that as } n \to \infty,
\]
\[
(u_{i,j}^i, v_{i,j}^i) \to (u_{i,j}^0, v_{i,j}^0) \text{ strongly in } H, \text{ where } \left(\begin{array}{c} u_{i,j}^0 \\ v_{i,j}^0 \end{array}\right) \in N_{i,j}(\varepsilon).
\]
5 Our main results

**Theorem A.** Assume $\mu_1, \mu_2 > 0, \beta < 0$ and the potentials $V_1, V_2 \in C(\mathbb{R}^N)$ satisfy the conditions (I) and (II). Then

(i) There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, problem $(E_\varepsilon)$ has $\kappa \times m$ solutions $\{(u_{i,j}^{\varepsilon}, v_{i,j}^{\varepsilon}) : i = 1, \cdots, \kappa, j = 1, \cdots, m\}$ satisfying

$$\varepsilon^{-N} J_{\varepsilon} (u_{i,j}^{\varepsilon}, v_{i,j}^{\varepsilon}) < \alpha_{\lambda_1, \mu_1} + \alpha_{\lambda_2, \mu_2} + \delta_0,$$

for $i = 1, \cdots, \kappa$ and $j = 1, \cdots, m$, where $\delta_0$ is a positive constant independent of $\varepsilon$. 


(ii) Fix $i \in \{1, 2, \ldots, \kappa\}$ and $j \in \{1, 2, \ldots, m\}$ arbitrarily. Then $u_{\varepsilon}^{i,j}$ has a concentration point $P_{1,i,j}^{\varepsilon}$ and $v_{\varepsilon}^{i,j}$ has another concentration point $P_{2,i,j}^{\varepsilon}$ such that
\[
\varepsilon^{-N} J_{\varepsilon} (u_{\varepsilon}^{i,j}, v_{\varepsilon}^{i,j}) \to \alpha_{\lambda_1, \mu_1} + \alpha_{\lambda_2, \mu_2}, \quad (5.4)
\]
$P_{1,i,j}^{\varepsilon} \to a^i$, $P_{2,i,j}^{\varepsilon} \to b^j$ and $|P_{1,i,j}^{\varepsilon} - P_{2,i,j}^{\varepsilon}|/\varepsilon \to +\infty$ as $\varepsilon$ goes to zero (up to a subsequence).
**Theorem B.** Assume \( \mu_1, \mu_2 > 0, \beta < 0 \), the potentials \( V_1, V_2 \in C(\mathbb{R}^N) \) satisfy the conditions (I) and (II) and there exists \( 1 \leq k_0 \leq \min\{\kappa, m\} \) such that \( a^j = b^i \) for all \( i = 1, 2, \ldots, k_0 \). Then there exists \( \varepsilon_* > 0 \) such that for \( \varepsilon < \varepsilon_* \), the Problem \((E_\varepsilon)\) has at least \( \kappa \times m + k_0 \) positive solutions.
Theorem C. Assume $\mu_1, \mu_2 > 0, \beta < 0$, the potentials $V_1, V_2 \in C(\mathbb{R}^N)$ satisfy the conditions (I) and (II). Then

(i) if $\liminf_{|x| \to \infty} V_l(x) \equiv V_{l,\infty} > \lambda_l$ for all $l = 1, 2$, then there exists $\varepsilon_\ast > 0$ such that for every $\varepsilon < \varepsilon_\ast$, we can find at least one least energy solution in the these solutions of Theorems A, B;

(ii) if $\lim_{|x| \to \infty} V_l(x) = \lambda_l$ for all $l = 1, 2$, then the all solutions of Theorems A, B are higher energy solutions.