Multiple positive solutions of semilinear elliptic boundary value problems in infinite strips

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Outline

- History and motivation
- The decomposition of Nachri manifold and main results
- Conclusions
1 History and motivation

Ambrosetti-Brezis-Cerami [J. Funct. Anal., 1994], consider the semilinear elliptic equation involving concave and convex nonlinearities

\[
\begin{cases}
-\Delta u = \lambda |u|^{q-2} u + |u|^{p-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(\(E_\lambda\))

where

(i) \(\lambda > 0\),

(ii) \(1 < q < 2 < p \leq 2^* \left( 2^* = \frac{2N}{N-2}, \text{ if } N \geq 3; 2^* = \infty, \text{ if } N = 2 \right)\),

(iii) \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^N\).
- There exists $\lambda_0 > 0$ such that

  - Eq. $(E_\lambda)$ has at least two positive solutions for $\lambda < \lambda_0$,
  - Eq. $(E_\lambda)$ has a positive solution for $\lambda = \lambda_0$,
  - Eq. $(E_\lambda)$ does not admit any positive solution for $\lambda > \lambda_0$.

- If $\Omega$ is a starshaped domain and $p = 2^*$, then for any sequence
  \(\{\lambda_n\} \subset \mathbb{R}^+\) with $\lambda_n \searrow 0$ as $n \to \infty$, there exists a sequence
  \(\{u_{\lambda_n}\}\) of positive solutions of Eq. $(E_{\lambda_n})$ such that

  \[||u_{\lambda_n}||_\infty \to \infty\] as $n \to \infty$. 

When $\Omega = B^N (0 ; 1) = \left\{ x \in \mathbb{R}^N : |x| < 1 \right\}$.

- There exists $\lambda_0 > 0$ such that
  
  - Eq. $(E_\lambda)$ has exactly two positive solutions for $\lambda < \lambda_0$,  
  - Eq. $(E_\lambda)$ has exactly one positive solution for $\lambda = \lambda_0$,  
  - Eq. $(E_\lambda)$ does not admit any positive solution for $\lambda > \lambda_0$.

1. Adimurthi-Pacella-Yadava [Diff. Int. Eqns, 1997],

2. Damascelli-Grossi-Pacella [Annls Inst. H. Poincaré Analyse Non lineairé, 1999],

3. Ouyang and Shi [J. Diff. Eqns., 1999],

- Eq. \((E_\lambda)\) in \(\Omega = B^N(0; 1)\)

**Fig. 1 :** \(2 < p < 2^*\)

**Fig. 2 :** \(p = 2^*\)
Extended Equations

The equations involving sign-changing weight functions

\[
\begin{cases}
-\Delta u = \lambda a(x) |u|^{q-2} u + b(x) |u|^{p-2} u & \text{in } \Omega, \\
u = 0 & \text{in } \partial \Omega,
\end{cases}
\]

\((E_{a,b})\)

where

(i) \( \lambda > 0 \),

(ii) \( 1 < q < 2 < p \leq 2^* \),

(iii) \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \),

(iv) \( a \in L^{r_q} (\Omega) \) and \( a^+ = \max \{a, 0\} \neq 0 \) where \( r_q = \frac{r}{r-q} \) for some \( r \in (q, 2^*) \);

(v) \( b \in L^{s_p} (\Omega) \) and \( b^+ = \max \{b, 0\} \neq 0 \) where \( s_p = \frac{s}{s-p} \) for some \( s \in (p, 2^*) \).
• de Figueiredo-Gossez-Ubilla [J. Funct. Anal., 2003](Sub-super solution) proved that

(i) If $p < 2^*$ and the weight functions $a, b$ are satisfying the following conditions:

(C1) $\exists$ nonempty open subset $\Omega_1 \subset \Omega$ s.t., on $\Omega_1$, $a(x) \geq \varepsilon_1$ for some $\varepsilon_1 > 0$ and $b(x)$ is bounded from below;

(C2) $\exists$ nonempty open subset $\Omega_2 \subset \Omega$ s.t., on $\Omega_2$, $b(x) \geq \varepsilon_2$ for some $\varepsilon_2 > 0$ and $a(x)$ is bounded from below,

then there exists $\lambda_* > 0$ such that Eq. $(E_{a,b})$ has at least two nontrivial nonnegative solutions for $\lambda < \lambda_*$. 

(ii) If $p \leq 2^*$, then there exists $\lambda^* > 0$ such that Eq. $(E_{a,b})$ does not admit any nontrivial nonnegative solution for $\lambda > \lambda^*$. 

• Brown-Wu [Elect. J. Diff. Eqns., 2007] (Fibering method) proved that

If $p < 2^*$, then there exists $\lambda_* > 0$ such that Eq. $(E_{a,b})$ has at least two positive solutions for $\lambda < \lambda_*$. 
  (Sub-super solution) proved that

If $p = 2^*$ and the weight functions $a \geq 0$ and $b \equiv 1$, then there exists $\lambda_0 > 0$ such that Eq. $(E_{a,b})$ has at least two nontrivial nonnegative solutions for $\lambda < \lambda_0$. 

(i) If $p = 2^*$ and the weight function $b \equiv 1$, then $\exists \lambda_0 > 0$ such that Eq. $(E_{a,b})$ has at least two positive solutions for $\lambda < \lambda_0$;

(ii) For any sequence $\{\lambda_n\} \subset \mathbb{R}^+$ with $\lambda_n \not\to 0$ as $n \to \infty$, there exists a sequence $\{u_{\lambda_n}\}$ of positive solutions of Eq. $(E_{a,b})$ such that

$$\|u_{\lambda_n}\|_\infty \to \infty \text{ as } n \to \infty.$$
Eq. $(E_{a,b})$ in bounded domains

Fig. 3: $2 < p < 2^*$

Fig. 4: $p = 2^*$
Problems

Semilinear elliptic equation involving concave and convex nonlinearities

\[
\begin{cases}
-\Delta u + \mu u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{p-2}u & \text{in } S, \\
u = 0 & \text{on } \partial S,
\end{cases}
\]

where

(i) \( \mu \geq 0 \) and \( \lambda > 0 \),

(ii) \( 1 \leq q < 2 < p < 2^* \),

(iii) \( S = \omega \times \mathbb{R} \) is an infinite strip, where \( \omega \) is a smooth bounded domain in \( \mathbb{R}^{N-1} \).
Let $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, we assume that $f(x)$ and $g(x)$ satisfy

$(f1)$ $f \in L_H(S)$, where $L_H(S) = L^{\frac{2}{2-q}}(S)$ if $1 < q < 2$ and $L_H(S) = H^{-1}(S)$ if $q = 1$;

$(f2)$ $f(x) \geq 0$ for all $x \in S$,

$(g1)$ $g(x) \in C(S)$ and $g(x) \to 1$ as $|x_N| \to \infty$;

$(g2)$ there exist $\delta > \theta_1$ and $0 < C_0 < 1$ such that

$$g(x) \geq 1 - C_0 \exp\left(-2\sqrt{1+\delta |x_N|}\right)$$

for all $x = (x', x_N) \in S$,

where $\theta_1$ is the first eigenvalue of the Dirichlet problem $-\Delta \phi = \theta \phi$ in $\Theta$, $\phi = 0$ on $\partial \Theta$. 
• Are there exist at least two positive solutions for equation \((E_{\lambda f, g})\) if \(\lambda\) is sufficiently small?
2 The decomposition of Nehari manifold and main results

The solutions of Eq. \((E_{\lambda f, g})\) are the same as the critical points of the energy functional \(J_\lambda : H^1_0(S) \to \mathbb{R},\)

\[
J_\lambda (u) = \frac{1}{2} \|u\|^2_{H^1} - \frac{\lambda}{q} \int_S f(x) |u|^q \, dx - \frac{1}{p} \int_S g(x) |u|^p \, dx,
\]

where \(H^1_0(S)\) is the Sobolev space with standard norm

\[
\left( \int_S |\nabla u|^2 + \mu u^2 \, dx \right)^{1/2}.
\]

- \(J_\lambda \in C^1 \left( H^1_0(S), \mathbb{R} \right),\)

- \(u\) is the critical point of \(J_\lambda\) in \(H^1_0(S),\) if \(J'_\lambda (u) = 0\) in \(H^{-1}(S).\)
From Nehari [Trans. Amer. Math. Soc., 1960], we know the Nehari manifold

$$M_\lambda = \{ u \in H_0^1 (S) \setminus \{0\} \mid \langle J'_\lambda (u), u \rangle = 0 \},$$

where $\langle , \rangle$ denotes the usual duality between $H_0^1 (S)$ and $H^{-1} (S)$.

- All nonzero critical points of $J_\lambda$ must lie on $M_\lambda$,
- $u \in M_\lambda$ if and only if

$$\| u \|_{H^1}^2 - \lambda \int_S f (x) |u|^q \, dx - \int_S g (x) |u|^p \, dx = 0.$$
Define a map \( \psi_\lambda : H_0^1 (S) \setminus \{0\} \to \mathbb{R} \) is given by

\[
\psi_\lambda (u) = \langle J'_\lambda (u) , u \rangle = \| u \|^2_{H^1} - \lambda \int_S f(x) |u|^q \, dx - \int_S g(x) |u|^p \, dx.
\]

Then

- \( u \in M_\lambda \) if and only if \( \psi_\lambda (u) = 0 \),
- for \( u \in M_\lambda \),

\[
\langle \psi'_\lambda (u) , u \rangle = 2 \| u \|^2_{H^1} - \lambda q \int_S f(x) |u|^q \, dx - p \int_S g(x) |u|^p \, dx,
\]
Hence if we define

\[ M^{+}_\lambda = \{ u \in M_\lambda : \langle \psi'_\lambda (u), u \rangle > 0 \}, \]
\[ M^0_\lambda = \{ u \in M_\lambda : \langle \psi'_\lambda (u), u \rangle = 0 \}, \]
\[ M^-_\lambda = \{ u \in M_\lambda : \langle \psi'_\lambda (u), u \rangle < 0 \}. \]

Then we have

- if \( \lambda = 0 \), then \( M^+_0 \) and \( M^0_0 \) are empty (i.e. \( M_0 = M^-_0 (S) \));
- there exists \( C \left( \lambda, q, p, \| a \|_{L^{\frac{2}{2-q}}} \right) > 0 \) such that
  \( \| u \|_{H^1} \leq C \left( \lambda, q, p, \| a \|_{L^{\frac{2}{2-q}}} \right) \) for all \( u \in M^0_\lambda \);
• if $u$ is a local minimizer for $J_\lambda$ in $\mathbf{M}_\lambda$ and $u \notin \mathbf{M}_\lambda^0$, then
  $J'_\lambda(u) = 0$ in $H^{-1}(\mathbf{S})$ (see Brown-Zhang [J. Diff. Eqns., 2003]);

• There exists $\lambda_1 > 0$ such that $\mathbf{M}_\lambda^0 = \emptyset$ for all $\lambda < \lambda_1$ (i.e.
  $\mathbf{M}_\lambda = \mathbf{M}_\lambda^+ \cup \mathbf{M}_\lambda^-$). (see Brown-Wug [Electr. J. Diff. Eqns.,
  2007]).

Define

$$
\theta_\lambda^+ = \inf_{u \in \mathbf{M}_\lambda^+} J_\lambda(u); \quad \theta_\lambda^- = \inf_{u \in \mathbf{M}_\lambda^-} J_\lambda(u).
$$

Then we have

• There exists $\lambda_2 \in (0, \lambda_1]$ such that if $\lambda < \lambda_2$, then $\theta_\lambda^+ < 0$ and
  $\theta_\lambda^- > c_0$ for some $c_0 > 0$;
• There exist minimizing sequences \( \{u_n^\pm\} \subset M_{\lambda}^\pm \) such that

\[
J_{\lambda}(u_n^\pm) = \theta_{\lambda}^\pm + o(1) \quad \text{and} \quad J'_{\lambda}(u_n^\pm) = o(1) \quad \text{in} \quad H^{-1}(S).
\]

(by the Ekeland variational principle)

• For each sequence \( \{u_n\} \subset M_{\lambda} \) which satisfies

\[
J_{\lambda}(u_n) = \sigma + o(1) \quad \text{with} \quad \sigma < \theta_{\lambda}^+ + \theta_0^-
\]

and

\[
J'_{\lambda}(u_n) = o(1) \quad \text{in} \quad H^{-1}(S)
\]

has a convergent subsequence.
Theorem 2.1  Suppose that the conditions (f1) – (f2) and (g1) – (g2) hold. Then for \( \lambda < \lambda_2 \), equation \((E_{\lambda f,g})\) possesses at least two positive solutions.
\[ g(x) \leq 1 \text{ with a strict inequality on a set of positive measure} \]

First we consider the filtration of the submanifold \( M_\lambda^- \) as follows:

\[
N_\lambda(\delta) = \{ u \in M_\lambda^- \mid J_\lambda(u) \leq \theta_0^- + \delta \}
\]

Then we have the following results.

**Theorem 2.2** There exist positive numbers \( \lambda_3 \leq \lambda_2 \) and \( \delta_0 \) such that for every \( \lambda < \lambda_3 \) and \( \delta < \delta_0 \) there exist two subsets \( N_{\lambda,1}(\delta) \) and \( N_{\lambda,2}(\delta) \) of \( N_\lambda(\delta) \) such that

(i) \( N_{\lambda,i}(\delta) \neq \emptyset \);

(ii) \( N_{\lambda,1}(\delta) \cap N_{\lambda,2}(\delta) = \emptyset \);

(iii) \( N_\lambda(\delta) = N_{\lambda,1}(\delta) \cup N_{\lambda,2}(\delta) \).
Define

$$\theta_{\lambda,1}^- = \inf_{u \in N_{\lambda,1}(\delta)} J_\lambda(u); \ \theta_{\lambda,2}^- = \inf_{u \in N_{\lambda,2}(\delta)} J_\lambda(u).$$

Then by the Ekeland variational principle

- There exist minimizing sequences \( \{u_n^{(i)}\} \subset N_{\lambda,i}(\delta) \) such that

$$J_\lambda \left( u_n^{(i)} \right) = \theta_{\lambda,i}^- + o(1) \quad \text{and} \quad J'_\lambda \left( u_n^{(i)} \right) = o(1) \quad \text{in} \ H^{-1}(S).$$
Theorem 2.3 If in addition to the conditions \((f1) - (f2)\) and 
\((g1) - (g2)\), we still have 
\((g3) \ g(x) \leq 1\) on \(S\) with a strict inequality on a set of positive
measure. 
Then for \(\lambda < \lambda_3\), equation \((E_{\lambda f,g})\) possesses at least three positive
solutions.
3 Conclusions

• Suppose that the conditions $(f1) - (f2)$ and $(g1) - (g2)$ hold. Then for $\lambda < \lambda_2$, equation $(E_{\lambda f, g})$ possesses at least two positive solutions.

• If in addition to the conditions $(f1) - (f2)$ and $(g1) - (g2)$, we still have $(g3) \ g(x) \leq 1$ on $S$ with a strict inequality on a set of positive measure. Then for $\lambda < \lambda_3$, equation $(E_{\lambda f, g})$ possesses at least three positive solutions.