Chapter 1

Functions and Limits

1.1 Functions

Definition 1 (Real-Valued Function of a Real Variable)(i) Let X and Y be sets of real numbers. A Real-Valued function $f$ of a real variable $x$ (independent variable) from $X$ to $Y$ is a correspondence that assigns to each number $x$ in $X$ exactly one number $y$ (dependent variable) in $Y$.

(ii) The domain of $f$ is the set $X$. The number $y$ is the image of $x$ under $f$ and is denoted by $f(x)$. The range of $f$ is a subset of $Y$ and consists of all images of numbers in $X$.

Example 1 (i) Equation in implicit form:

$$x^2 + 2y = 1;$$

(ii) Equation in explicit form

$$y = \frac{1}{2} (1 - x^2);$$

(iii) Function notation

$$f(x) = \frac{1}{2} (1 - x^2).$$

The original equation, $x^2 + 2y = 1$, implicitly defines $y$ as a function of $x$. When you solve the equation for $y$, you are writing the equation in explicit form.
**The Domain and Range of a Function:**
The domain of a function can be described explicitly, or it may be described implicitly by an equation used to define the function. The implied domain is the set of all real number for which the equation is defined.

**Example 2** *The domain of the function*

\[ s(t) = \frac{8t + 10}{t + 2} \]

**Example 3** *Determine the domain and range of the function*

\[ f(x) = \begin{cases} 
1 - x, & \text{if } x < 1 \\
\sqrt{1 - x}, & \text{if } x \geq 1 
\end{cases} \]

**The Graph of a Function:**
The graph of the function \( y = f(x) \) consists of all points \( (x, f(x)) \), where \( x \) is in the domain of \( f \). Note that

(i) \( x = \) the directed distance from the \( y \)-axis;
(ii) \( f(x) = \) the the directed distance from the \( x \)-axis.

**Example 4**

\[ s(t) = \frac{8t + 10}{t + 2} \]

**Example 5**

\[ f(x) = \begin{cases} 
1 - x, & \text{if } x < 1 \\
\sqrt{1 - x}, & \text{if } x \geq 1 
\end{cases} \]
**Constant Functions:**

\[ f(x) \equiv \text{constant} \]

**Elementary functions:**

Elementary functions fail into three categories:

(i) Algebraic function (polynomial, radical, rational).

(ii) Trigonometric function (sine, cosine, tangent, and so on).

(iii) Exponential and logarithmic.

**Composite Functions:**

You can combine two (elementary) functions in yet another way, called composition. The resulting function is called a composite function.

**Definition 2**  
Let \( f \) and \( g \) be function. The function given by \( (f \circ g)(x) = f(g(x)) \) is called the composite of \( f \) with \( g \). The domain of \( f \circ g \) is the set of all \( x \) in the domain of \( g \) such that \( g(x) \) is in the domain of \( f \).

**Example 6**  
Given \( g(x) = \sqrt{x} \) and \( g(x) = 5 - x \), find \( g \circ f \).

**Example 7**  
Let \( f(x) = \frac{5}{(x^2 + 9)^3} \)

Find two functions \( g(x) \) and \( u(x) \) whose composite is \( f(x) \).  
\( g(x) = \frac{5}{x^2} \),  \( u(x) = x^2 + 9 \).

**Even and Odd functions:**

**Definition 3**  
(Even and Odd)  
(i) The function \( y = f(x) \) is even if \( f(x) = f(-x) \).

(ii) The function \( y = f(x) \) is odd if \( f(-x) = -f(x) \).

**Example 8**  
(a) \( f(x) = x^3 - x \),  
(b) \( g(x) = 1 + \cos x \).