Multiplicity of single-spike solutions for two coupled nonlinear Schrödinger equations

Tsung-fang Wu
Department of Applied Mathematics
National University of Kaohsiung, Kaohsiung, Taiwan

(joint work with Tai-Chia Lin)
Outline

• The History of BEC
• Motivation
• Nehari manifold
• Main result
1 The History of BEC
2 Motivation

We consider a two-component system of nonlinear Schrödinger equations with optical lattice potentials:

\[
\begin{aligned}
\left\{
\begin{array}{ll}
\frac{\hbar^2}{2m} \Delta \Phi_j - \tilde{V}_j(x) \Phi_j + \sum_{i=1}^{2} \tilde{\beta}_{ij} |\Phi_i|^2 \Phi_j = -i\hbar \partial_t \Phi_j & \text{for } x \in \Omega, \\
\Phi_j(x, t) = 0 & \text{for } x \in \partial \Omega, \\
t > 0, \ j = 1, 2,
\end{array}
\right.
\end{aligned}
\]

where \( \Omega \subseteq \mathbb{R}^N \), \( N = 2, 3 \) is the region for condensate dwelling, \( \Phi_j \)'s are corresponding condensate wave functions, \( \hbar \) is Planck constant, \( m \) is the atomic mass, \( \tilde{\beta}_{jj} = -(N_j - 1)U_{jj} > 0 \) and \( \tilde{\beta}_{12} = \tilde{\beta}_{21} = -N_2 U_{12} \).
Here each $N_j \geq 1$ is a fixed number of atoms in the hyperfine state $|j\rangle$, and $U_{ij} = 4\pi \frac{\hbar^2}{2m} a_{ij}$, where $a_{jj}$’s and $a_{12}$ are the intraspecies and interspecies scattering lengths. $\tilde{V}_j$ is the optical lattice potential for the $j$-th species and is a periodic function of spatial variables written as (see [Jaksch etc., Phys. Rev. Lett., 1998])

$$\tilde{V}_j(x) = \sum_{k=1}^{N} \tilde{v}_{j,k} \sin^2(l x_k) \quad \text{for } x = (x_1, \cdots, x_N) \in \Omega, j = 1, 2,$$

(2.2)

where $\tilde{v}_{j,k} \geq 0$ is the associated potential depth, $l = 2\pi/L$, and $L$ is the wavelength of the laser light.
Here, we study steady state bright solitons with the form
\[ \Phi_j = e^{i\tilde{\lambda}_j x/\hbar} u_j(x), \quad j = 1, 2 \]
called bound states of the system (2.1), where \( \tilde{\lambda}_j \)'s are positive constants and \( u_j \)'s satisfy
\[
\begin{cases}
\frac{\hbar^2}{2m} \Delta u_j(x) - \tilde{V}_j(x) u_j(x) + \sum_{i=1}^2 \tilde{\beta}_{ij} u_i^2(x) u_j(x) = \tilde{\lambda}_j u_j & \text{in } \Omega, \\
u_j > 0 & \text{in } \Omega, \\
u_j = 0 & \text{on } \partial\Omega.
\end{cases}
\]
(2.3)

Physically, \( u_j \)'s are the associated condensate amplitudes of the bound states \( \Phi_j \)'s, and \( \tilde{\lambda}_j \)'s are chemical potentials. Due to Feshbach resonance (see [Inouye etc., Nature, 1998]), we may set \( \tilde{\beta}_{jj} \)'s, \( \tilde{\lambda}_j \)'s and \( \tilde{\nu}_{j,k} \)'s to be very large quantities.
By rescaling and some simple assumptions, the problem (2.3) with very large \( \tilde{\beta}_{jj} \)'s, \( \tilde{\lambda}_j \)'s and \( \tilde{\nu}_{j,k} \)'s is equivalent to the following singularly perturbed problem:

\[
\begin{aligned}
\epsilon^2 \Delta u_j - \hat{V}_j(x) u_j + \sum_{i=1}^{2} \beta_{ij} u_i^2 u_j &= 0 & \text{in } \Omega, \\
u_j &> 0 & \text{in } \Omega, \\
u_j &= 0 & \text{on } \partial\Omega, \ j = 1, 2,
\end{aligned}
\]

(2.4)

where \( 0 < \epsilon \ll 1 \) is a small parameter, and \( \hat{V}_j \)'s are defined by

\[
\hat{V}_j(x) = \lambda_j + \sum_{k=1}^{N} \nu_{j,k} \sin^2(l x_k), \quad \text{for } x = (x_1, \cdots, x_N) \in \Omega. \quad (2.5)
\]

Here \( \lambda_j > 0, \nu_{j,k} \geq 0, l > 0, \) and \( \beta_{ij} \)'s are constants independent of \( \epsilon. \)
Without loss of generality, problem (2.4) can be generalized to the following problem:

\[
\begin{cases}
\epsilon^2 \Delta u - V_1(x) u + \mu_1 u^3 + \beta uv^2 = 0 & \text{in } \mathbb{R}^N, \\
\epsilon^2 \Delta v - V_2(x) v + \mu_2 v^3 + \beta u^2 v = 0 & \text{in } \mathbb{R}^N, \\
u > 0, v > 0 & \text{in } \mathbb{R}^N,
\end{cases}
\]

\((E_{\epsilon})\)

where \(N = 1, 2, 3\), and the potentials \(V_1, V_2 \in C\left(\mathbb{R}^N\right)\) satisfy

(I) \(V_1\) and \(V_2\) are bounded functions on \(\mathbb{R}^N\) and
\[\Lambda_l = \sup_{x \in \mathbb{R}^N} |V_l(x)| < \infty, \ l = 1, 2.\]

(II) The global minima of potentials \(V_l, l = 1, 2\) only occur at \(k\) points \(a^1, a^2, \ldots, a^k \in \mathbb{R}^N\) if \(l = 1\), and \(m\) points \(b^1, b^2, \ldots, b^m \in \mathbb{R}^N\) if \(l = 2\), respectively.
Moreover,

\[ V_1 (a^i) = \lambda_1 \equiv \min \{ V_1 (x) : x \in \mathbb{R}^N \} > 0, \quad i = 1, 2, \ldots, \kappa, \]

and

\[ V_2 (b^j) = \lambda_2 \equiv \min \{ V_2 (x) : x \in \mathbb{R}^N \} > 0, \quad j = 1, 2, \ldots, m. \]
When the potentials $V_i$’s satisfy

$$0 < \lambda_l = \inf_{x \in \mathbb{R}^N} V_i(x) < \lim_{|x| \to \infty} V_i(x) \leq \infty \quad \text{for } l = 1, 2,$$

(2.6)

Lin and Wei study the minimization of the functional $J_\varepsilon$ over the manifold $\mathbb{N}_\varepsilon$. As $\beta < 0$ and $\varepsilon > 0$ sufficiently small, there exists a least energy solution $(u_{\varepsilon,1}, u_{\varepsilon,2})$ of problem $(E_\varepsilon)$ such that

$$\varepsilon^{-N} J_\varepsilon(u_{\varepsilon,1}, u_{\varepsilon,2}) \to \alpha \lambda_1, \mu_1 + \alpha \lambda_2, \mu_2,$$

(2.7)

as $\varepsilon$ goes to zero (up to a subsequence).
Remark: The our conditions (I) and (II) may allow the case that

\[
\liminf_{|x| \to \infty} V_l(x) < \limsup_{|x| \to \infty} V_l(x) \quad \text{for some } l = 1, 2.
\]

Hence the argument of Lin-Wei may not be applicable to problem \((E_\varepsilon)\) with the conditions (I) and (II).
Problems

To see how the potentials act on the existence and number of steady state bright solitons (i.e. ground states or bound states) when \( \kappa = m = 1 \) and \( a_i = b_j \).

i.e. we study the existence and multiplicity of one-spiky solutions of problem (2.4).

3 Nehari manifold

Let \((u, v) \in H\) be a solution of problem \((E_\varepsilon)\). Then

\[
\begin{align*}
\int_{\mathbb{R}^N} \varepsilon^2 \nabla u \nabla \varphi_1 + \int_{\mathbb{R}^N} V_1 u \varphi_1 + \int_{\mathbb{R}^N} \varepsilon^2 \nabla v \nabla \varphi_2 + \int_{\mathbb{R}^N} V_2 v \varphi_2 - \int_{\mathbb{R}^N} \mu_1 u^3 \varphi_1 \\
- \int_{\mathbb{R}^N} \mu_2 v^3 \varphi_2 - \beta \int_{\mathbb{R}^N} u v^2 \varphi_1 - \beta \int_{\mathbb{R}^N} u^2 v \varphi_2 = 0, \ \forall (\varphi_1, \varphi_2) \in H,
\end{align*}
\]

where \(H = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\) is a Sobolev space with the norm \(\| \cdot \|_H\) given by

\[
\|(\phi, \psi)\|_H^2 = \|\phi\|_{V_1}^2 + \|\psi\|_{V_2}^2
\]

\[
= \int_{\mathbb{R}^N} \left(\varepsilon^2 |\nabla \phi|^2 + V_1 \phi^2\right) + \int_{\mathbb{R}^N} \left(\varepsilon^2 |\nabla \psi|^2 + V_2 \psi^2\right).
\]
It is well known that the solution \((u,v) \in H\) of problem \((E_\varepsilon)\) is a critical point of the energy functional \(J_\varepsilon \in C^1(H, \mathbb{R})\) defined by

\[
J_\varepsilon (\phi, \psi) = \frac{1}{2} \|(\phi, \psi)\|^2_H - \frac{1}{4} \left( \int_{\mathbb{R}^N} \mu_1 \phi^4 + \int_{\mathbb{R}^N} \mu_2 \psi^4 \right) - \frac{\beta}{2} \int_{\mathbb{R}^N} \phi^2 \psi^2
\]

(3.1)

for all \((\phi, \psi) \in H\).

Note that the energy functional \(J_\varepsilon\) is not bounded below on \(H\), it is useful to consider the functional on the Nehari manifold
The type of Nehari manifold

A. codimension one

\[ \tilde{N}_\varepsilon = \{ (u, v) \in H \setminus \{(0, 0)\} : \langle J'_\varepsilon (u, v), (u, v) \rangle = 0 \} \]

where

\[ \langle J'_\varepsilon (u, v), (u, v) \rangle = \|(u, v)\|_H^2 - \left( \mu_1 \int_{\mathbb{R}^N} u^4 + \mu_2 \int_{\mathbb{R}^N} v^4 \right) - 2\beta \int_{\mathbb{R}^N} u^2 v^2. \]

- trivial solution \((0, 0) \notin \tilde{N}_\varepsilon\)
- semitrivial solutions \((u, 0), (0, v) \in \tilde{N}_\varepsilon\)
- nontrivial solution \((u, v) \in \tilde{N}_\varepsilon\)
B. codimension two

\[ \mathbf{N}_\varepsilon = \left\{ (u, v) \in \mathbf{T} : \begin{align*} \|u\|^2_{V_1} &= \mu_1 \int_{\mathbb{R}^N} u^4 + \beta \int_{\mathbb{R}^N} u^2 v^2 \\ \|v\|^2_{V_2} &= \mu_2 \int_{\mathbb{R}^N} v^4 + \beta \int_{\mathbb{R}^N} u^2 v^2 \end{align*} \right\} , \]

where \( \mathbf{T} = \{ (u, v) \in H : u \not\equiv 0 \text{ and } v \not\equiv 0 \} \).

- trivial solution \((0, 0) \notin \mathbf{N}_\varepsilon\)
- nontrivial solution \((u, v) \in \mathbf{N}_\varepsilon\)
Example: Let $V_l \equiv \lambda_l = 1 = \mu_l$ for $l = 1, 2$ and $\beta \in \mathbb{R}$. Define $\alpha_{\lambda_l, \mu_l}$ is the minimizing energy of the functional $I_{\lambda_l, \mu_l}$ over the Nehari manifold $M_{\lambda_l, \mu_l}$ defined by

$$\alpha_{\lambda_l, \mu_l} = \inf \{ I_{\lambda_l, \mu_l} (u) \mid u \in M_{\lambda_l, \mu_l} \} ,$$

$$I_{\lambda_l, \mu_l} (u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u|^2 + \lambda_l u^2 \right) - \frac{\mu_l}{4} \int_{\mathbb{R}^N} u^4 , \quad \forall : u \in H^1 (\mathbb{R}^N)$$

$$M_{\lambda_l, \mu_l} = \{ u \in H^1 (\mathbb{R}^N) \setminus \{0\} \mid \langle I'_{\lambda_l, \mu_l} (u), u \rangle = 0 \} ,$$

for $l = 1, 2$. 
\[ \theta(\beta) = \inf_{(u,v) \in \mathbb{N}_e} J_{\epsilon}(u,v) \begin{cases} \leq \frac{2}{1+\beta} \alpha_{1,1}, & \text{if } 0 \leq \beta \leq 1 \\ = 2\alpha_{1,1}, & \text{if } \beta \leq 0 \end{cases} \]

\[ \tilde{\theta}(\beta) = \inf_{(u,v) \in \mathbb{N}_e} J_{\epsilon}(u,v) \leq \frac{2}{1+\beta} \alpha_{1,1}, \text{if } \beta \geq 1 \]
Thus we should choose the Nehari manifold with codimension two.

**Lemma 1.** Let \((u_0, v_0)\) be a constrained critical point of \(J_\varepsilon\) on \(N_\varepsilon\) with

\[
\mu_1 \mu_2 \int_{\mathbb{R}^N} u_0^4 \int_{\mathbb{R}^N} v_0^4 - \beta^2 \left( \int_{\mathbb{R}^N} u_0^2 v_0^2 \right)^2 > 0.
\]

Then \(\nabla J_\varepsilon (u_0, v_0) = 0\) on \(H^*\).

**proof:** Define

\[
f (u, v) = \|u\|_{V_1}^2 - \mu_1 \int_{\mathbb{R}^N} u^4 - \beta \int_{\mathbb{R}^N} u^2 v^2,
\]

\[
g (u, v) = \|v\|_{V_2}^2 - \mu_2 \int_{\mathbb{R}^N} v^4 - \beta \int_{\mathbb{R}^N} u^2 v^2.
\]

Since \((u_0, v_0)\) is a constrained critical point of \(J_\varepsilon\) on \(N_\varepsilon\), by the Lagrange multiplier, there exist \(\theta_1, \theta_2 \in \mathbb{R}\) such that

\[
\nabla J_\varepsilon (u_0, v_0) = \theta_1 \nabla f (u_0, v_0) + \theta_2 \nabla g (u_0, v_0).
\]
Then we have
\[
\theta_1 \left[ \|u_0\|_{V_1}^2 - 2\mu_1 \int_{\mathbb{R}^N} u_0^4 - 2\beta \int_{\mathbb{R}^N} u_0^2 v_0^2 \right] - \theta_2 \left[ \beta \int_{\mathbb{R}^N} u_0^2 v_0^2 \right] = 0,
\]
\[
-\theta_1 \left[ \beta \int_{\mathbb{R}^N} u_0^2 v_0^2 \right] + \theta_2 \left[ \|v_0\|_{V_2}^2 - 2\mu_2 \int_{\mathbb{R}^N} v_0^4 - 2\beta \int_{\mathbb{R}^N} u_0^2 v_0^2 \right] = 0
\]
or
\[
\Sigma \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
where
\[
\Sigma = \begin{bmatrix}
\mu_1 \int_{\mathbb{R}^N} u_0^4 & \beta \int_{\mathbb{R}^N} u_0^2 v_0^2 \\
\beta \int_{\mathbb{R}^N} u_0^2 v_0^2 & \mu_2 \int_{\mathbb{R}^N} v_0^4
\end{bmatrix}.
\]
Then
\[
\det \Sigma = \mu_1 \mu_2 \int_{\mathbb{R}^N} u_0^4 \int_{\mathbb{R}^N} v_0^4 - \beta^2 \left( \int_{\mathbb{R}^N} u_0^2 v_0^2 \right)^2 > 0,
\]
this implies \( \theta_1 = \theta_2 = 0. \) Therefore \( \nabla J_\varepsilon (u_0, v_0) = 0 \) on \( \mathcal{H}^* \)
Now we state results which are useful to prove our main result as follows:

**Lemma 2.** Let $\beta < 0$, $\varepsilon, \sigma > 0$ and $(u, v) \in \mathbb{N}_\varepsilon$. If
\[
\int_{\mathbb{R}^N} u^2v^2 \geq \sigma \varepsilon^N,
\]
then
\[
\varepsilon^{-N} J_\varepsilon (u, v) > \alpha \lambda_{1,\mu_1} + \alpha \lambda_{2,\mu_2} + \bar{\delta}
\]  
(3.2)

where
\[
\bar{\delta} = \frac{|\beta| \sigma}{2} \min \left\{ 1, \frac{1}{2} \min \left\{ \frac{S_1^2}{\mu_1 (4\alpha \lambda_{1,\mu_1} + |\beta| \sigma)}, \frac{S_2^2}{\mu_2 (4\alpha \lambda_{2,\mu_2} + |\beta| \sigma)} \right\} \right\}
\]  
(3.3)

and
\[
S_i = \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \left| \nabla u \right|^2 + \lambda_i u^2}{\left( \int_{\mathbb{R}^N} u^4 \right)^{1/2}} > 0, \quad i = 1, 2.
\]  
(3.4)
Furthermore, if \( 0 < \sigma < \frac{4\sqrt{\alpha_{\lambda_1, \mu_1} \alpha_{\lambda_2, \mu_2}}}{|\beta|} \), then

\[
\left[ \mu_1 \mu_2 \int_{\mathbb{R}^N} u^4 \int_{\mathbb{R}^N} v^4 - \beta^2 \left( \int_{\mathbb{R}^N} u^2 v^2 \right)^2 \right] > \varepsilon^{2N} \left( 16\alpha_{\lambda_1, \mu_1} \alpha_{\lambda_2, \mu_2} - \beta^2 \sigma^2 \right) > 0
\]

for all \( (u, v) \in N_\varepsilon \) and \( \varepsilon^{-N} J_\varepsilon (u, v) \leq \alpha_{\lambda_1, \mu_1} + \alpha_{\lambda_2, \mu_2} + \overline{\delta} \).
4 Main result

**Definition** (Lusternik-Schnirelman category):

(i) For a topological space $X$, we say a non-empty, closed subset $Y \subset X$ is contractible to a point in $X$ iff there exists a continuous mapping

$$\xi : [0, 1] \times Y \rightarrow X$$

such that for some $x_0 \in X$

$$\xi (0, x) = x \text{ for all } x \in Y,$$

and

$$\xi (1, x) = x_0 \text{ for all } x \in Y.$$
(ii) We define

\[ \text{cat}(X) = \min \{k \in \mathbb{N} \mid \text{there exist closed subsets } Y_1, \ldots, Y_k \subset X \text{ s.t.} \]

\[ Y_j \text{ is contractible to a point in } X \forall j \text{ and } \bigcup_{j=1}^{k} Y_j = X \} . \]
**Lemma 3** (Ambrosetti, 1992) Suppose that $X$ is a Hilbert manifold and $F \in C^1(X, \mathbb{R})$. Assume that there are $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$,

(i) $F(x)$ satisfies the Palais–Smale condition for energy level $c \leq c_0$;
(ii) $\text{cat} (\{x \in X \mid F(x) \leq c_0\}) \geq k$.

Then $F(x)$ has at least $k$ critical points in $\{x \in X ; F(x) \leq c_0\}$.

**Lemma 4** (Adachi–Tanaka, 2000) Let $X$ be a topological space. Suppose that there are two continuous maps

$$\Phi : \mathbb{S}^{N-1} \to X, \quad \Psi : X \to \mathbb{S}^{N-1}$$

such that $\Psi \circ \Phi$ is homotopic to the identity map of $\mathbb{S}^{N-1}$. Then

$$\text{cat} (X) \geq 2.$$
We have the Palais–Smale condition of $J_\varepsilon$ as follows:

**Proposition.** Assume $\{(u_n, v_n)\}_{n=1}^\infty$ is a sequence in $N_\varepsilon$ satisfying

(i) $J_\varepsilon(u_n, v_n) \to \varepsilon^N \theta$ as $n \to \infty$ where $\theta < \alpha_{\lambda_1, \mu_1} + \alpha_{\lambda_2, \mu_2} + \delta$;

(ii) $J'_\varepsilon(u_n, v_n) \to 0$ strongly in $H^*$ as $n \to \infty$.

Then there exists a subsequence such that $(u_n, v_n) \to (u_0, v_0)$ strongly in $H$, where $(u_0, v_0) \in N_\varepsilon$. 
For $c \in \mathbb{R}^+$, we denote

$$\left[ \varepsilon^{-N} J_\varepsilon \leq c \right] = \{(u, v) \in \mathbb{N}_\varepsilon \mid \varepsilon^{-N} J_\varepsilon (u, v) \leq c \}.$$ 

Then we have the following result.

**Theorem.** There exists a positive number $\varepsilon_*$ such that for $0 < \varepsilon < \varepsilon_*$,

$$\text{cat} \left( \left[ \varepsilon^{-N} J_\varepsilon \leq \alpha \lambda_{1,1} + \alpha \lambda_{2,2} + \overline{\delta} \right] \right) \geq 2,$$