Multiple Positive Solutions of Nonhomogeneous Elliptic Equations in Exterior Domains

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Abstract
In this paper, we consider the following nonhomogeneous elliptic problem
\[
\begin{cases}
-\Delta u + u = \lambda (u^{p-1} + h(x)) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u \in H^1_0(\Omega),
\end{cases}
\]
where \(2 < p < \frac{2N}{N-2}\) (\(N \geq 3\)), \(\lambda > 0\), \(\Omega = \mathbb{R}^N\) is an exterior domain in \(\mathbb{R}^N\) where \(\omega\) is a bounded set with smooth boundary, \(h \in L^2(\Omega) \cap L^\beta(\Omega)\) \((\beta > \frac{N}{2} \text{ if } N \geq 4, \beta = \frac{N}{2} \text{ if } N = 3)\) is nonnegative and \(h(x) \not\equiv 0\). We use variational methods to show that there exists a positive number \(\lambda_0\) such that the equation \((E_\lambda)\) has at least two positive solutions if \(\lambda \in (0, \lambda_0)\), no positive solution if \(\lambda > \lambda_0\) and at least one positive solution if \(\lambda = \lambda_0\). Furthermore, we use the Lusternik-Schnirelman category to show that there exists a positive number \(\lambda_* < \lambda_0\) such that the equation \((E_\lambda)\) has at least three positive solutions if \(\lambda \in (0, \lambda_*)\).

1 Introduction
In this paper, we study the existence and multiplicity of positive solutions for the following nonhomogeneous elliptic equation:
\[
\begin{cases}
-\Delta u + u = \lambda (u^{p-1} + h(x)) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u \in H^1_0(\Omega),
\end{cases}
\]

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where $2 < p < \frac{2N}{N-2}$ ($N \geq 3$), $\lambda > 0$, $\Omega = \mathbb{R}^N$ where $\omega$ is a bounded set with smooth boundary, $h \in L^2(\Omega) \cap L^2(\Omega)$ ($\beta > \frac{N}{2}$ if $N \geq 4$, $\beta = \frac{N}{2}$ if $N = 3$) is nonnegative and $h(x) \not\equiv 0$. Associated with equation $(E_\lambda)$, we consider the energy functional:

$$J_\lambda(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + u^2) - \frac{\lambda}{p} \int_{\Omega} |u|^p - \lambda \int_{\Omega} h(x) u.$$ 

It is well known that the solutions of equation $(E_\lambda)$ are the critical points of the energy functional $J_\lambda$.

In solving the above equation $(E_\lambda)$ by variational methods, we find two kinds of difficulties: (i) the Laplacian from $H^2(\Omega)$ in $L^2(\Omega)$ is not Fredholm; (ii) the injection $H^1(\Omega)$ in $L^2(\Omega)$ is not compact. Indeed, although there exist various general methods to solve the analogous equation $(E_\lambda)$ when $\Omega$ is a bounded domain, (see Amann [3], Bahri-Berestycki [6], Bahri-Lions [7], Brézis-Nirenberg [11], Crandall-Rabinowitz [13], Lions [23], Rabinowitz [25] and Tarantello [27], etc.), these arguments break down in the above situation. To overcome these difficulties, we need to understand precisely how the loss of compactness for sequences of approximated solutions of equation $(E_\lambda)$ occurs by using the concentration-compactness method of Lions [24]. Such a question was studied by Zhu [32] and Zhu-Zhou [33]. Furthermore, Zhu-Zhou [33] proved that there is a $\lambda_0 > 0$ such that the equation $(E_\lambda)$ has at least two positive solutions if $\lambda \in (0, \lambda_0)$ and no positive solution if $\lambda > \lambda_0$. But it is still unknown whether $\lambda_0$ is bounded or infinite. In this paper, we can show that the positive number $\lambda_0$ is bounded. Furthermore, if we combine above and the results of Zhu-Zhou [33], then we can get the following results.

**Theorem 1.1** There exists a positive number $\lambda_0$ such that

(i) for all $\lambda \in (0, \lambda_0)$, the equation $(E_\lambda)$ has at least two positive solutions and we can find a minimal positive solution $u_\lambda$ such that $J_\lambda(u_\lambda) < 0$;

(ii) for $\lambda = \lambda_0$, the equation $(E_{\lambda_0})$ has at least one positive solution $u_0$;

(iii) for all $\lambda \in (\lambda_0, \infty)$, the equation $(E_\lambda)$ has no positive solution.

Henceforth we use the symbol $u_\lambda$ for $\lambda \in (0, \lambda_0)$ to denote the minimal positive solution of equation $(E_\lambda)$. The proof of Theorem 1.1, based on the method of subsolutions and supersolutions and considering the following minimization problem: for $v \in H^1_0(\Omega) \setminus \{0\}$ is nonnegative, let

$$\mu(v) = \inf \left\{ \int_{\Omega} |\nabla u|^2 + u^2 \mid u \in H^1_0(\Omega), \lambda (p-1) \int_{\Omega} u^{p-2} u^2 = 1 \right\}.$$ 

We show that if $u_\lambda$ is a minimal positive solution of equation $(E_\lambda)$, then $\mu(u_\lambda) > 1$. Moreover, we can prove some uniqueness results.
Theorem 1.2 Let $\lambda_0 > 0$ as in Theorem 1.1. Then for all $\lambda \in (0, \lambda_0)$

(i) the minimal positive solution $u_\lambda$ is the only positive solution satisfying $\mu (u_\lambda) > 1$;

(ii) if the equation $(E_\lambda)$ has a positive solution $v > u_\lambda$ with $\mu (v) < 1$, then there does not exist another positive solution $v'$ of equation $(E_\lambda)$ such that $u_\lambda < v < v'$.

By the change of variables $v (x) = \lambda^{1/(p-2)} u (x)$, the equation $(E_\lambda)$ is transformed to

$$
\begin{cases}
-\Delta v + v = v^{p-1} + \lambda^{\frac{p-1}{p-2}} h & \text{in } \Omega, \\
v > 0 & \text{in } \Omega, \\
v \in H^1_0 (\Omega).
\end{cases}
$$

(1)

Under the assumption $\lambda \neq 0$, our equation (1) can be regarded as a perturbation problem of the following homogeneous equation:

$$
\begin{cases}
-\Delta v + v = v^{p-1} & \text{in } \Omega, \\
v > 0 & \text{in } \Omega, \\
v \in H^1_0 (\Omega).
\end{cases}
$$

(2)

It is known that the existence of positive solutions of the homogeneous equation (2) is affected by the shape of the domain. This has been the focus of a great deal of research by several authors (see Ambrosetti-Rabinowitz [5], Bahri-Lions [8], Benci-Cerami [9], Berestycki-Lions [10], Esteban-Lions [17], Lions [24], Lien-Tzeng-Wang [21], Del Pino-Felmer [15, 16] and Wu [30, 31], etc.). Furthermore, if $\Omega = \mathbb{R}^N$, the equation (2) has a unique positive solution (see Kwong [20]).

For the equation (2) in exterior domain $\Omega$, we can see that the Mountain Pass value is equal to the first level of breaking down of Palais–Smale condition (see Benci-Cerami [9]) and we can not get a positive solution through the Mountain Pass Theorem (i.e. equation (2) does not admit any ground state solution). However, Benci-Cerami [9] showed the existence of at least one positive solution of equation (2) under the following condition:

$$
\omega \subset B^N (0; \rho) = \{ x \in \mathbb{R}^N \mid |x| < \rho \} \text{ and } \rho \text{ is sufficiently small.}
$$

Furthermore, the critical value of their solution is strictly greater than the first break down of the Palais–Smale condition.

From the above situation and a similar idea in Adachi-Tanaka [1] who consider the following equation:

$$
\begin{cases}
-\Delta u + u = a (x) u^{p-1} + h (x) & \text{in } \mathbb{R}^N, \\
u > 0 & \text{in } \mathbb{R}^N, \\
u \in H^1 (\mathbb{R}^N),
\end{cases}
$$

($E_{a,h}$)

where $a (x) \leq 1 = \lim_{|x| \to \infty} a (x)$ and $h (x) \in H^{-1} (\mathbb{R}^N) \setminus \{0\}$ is nonnegative. Using the equation ($E_{a,0}$) does not admit any ground state solution, they proved
that the equation \((E_{a,h})\) has at least three positive solutions under the assumption \(\|h\|_{H^{-1}}\) is sufficiently small. Furthermore, under the additional condition
\[
a(x) - 1 \geq -C(-(2 + \delta)|x|)
\]
for some \(\delta > 0, C > 0\), then the equation \((E_{a,h})\) has at least four positive solutions. Thus, the existence of more than two positive solutions for equation \((E_\lambda)\) is expected and so we have the following results.

**Theorem 1.3** There exists a positive number \(\lambda_* < \lambda_0\) such that for \(\lambda \in (0, \lambda_*)\), the equation \((E_\lambda)\) has at least three positive solutions.

**Corollary 1.4** Suppose that \(\omega = B^N(0; \rho_0)\) and \(h(x) = h(|x|)\) is radially symmetric. Then for all \(\lambda \in (0, \lambda_*)\), the equation \((E_\lambda)\) has at least one positive solution which is not radially symmetric.

**Proof.** Similar to the proof of Corollary 0.3 in Adachi-Tanaka [1].

When \(\lambda \equiv 1\), there have been some progress as follows: Lin-Wang-Wu [22] proved that the equation \((E_1)\) has at least three positive solutions if \(\|h\|_{L^2}\) is sufficiently small and \(h(x)\) decays faster than \(\exp(-c|x|)\). Zhu [32] and Hirano [18] were mainly concerned with \(\Omega = \mathbb{R}^N\), \(h \in L^2(\mathbb{R}^N) \setminus \{0\}\) is nonnegative, and showed that the equation \((E_1)\) has at least two positive solutions when \(\|h\|_{L^2}\) is sufficiently small and \(h(x)\) decays faster than \(\exp(-c|x|)\). Generalizations of the result of [32] and [18] were done by Cao-Zhou [14], Jeanjean [19] and Adachi-Tanaka [2]. In [2], [14] and [19], the general equations
\[
\begin{align*}
-\Delta u + u &= f(x, u) + h(x) \quad \text{in } \mathbb{R}^N, \\
0 &< u \in H^1(\mathbb{R}^N)
\end{align*}
\]
were studied where \(f\) satisfies suitable conditions and \(h(x) \in H^{-1}(\mathbb{R}^N) \setminus \{0\}\) is nonnegative and the existence of at least two positive solutions when \(\|h\|_{H^{-1}}\) sufficiently small was proved.

This paper is organized as follows. In section 2, we prove \(\lambda_0\) is bounded. In section 3, we prove the Theorem 1.2. In section 4, we prove the Theorem 1.3.

## 2 Boundedness of \(\lambda_0\)

In this section we prove \(\lambda_0 > 0\) is bounded. First, we recall some results in Zhu-Zhou [33].

**Lemma 2.1** For each \(r > 0\), there exists \(\tilde{\lambda} > 0\) such that for \(\lambda \in \left(0, \tilde{\lambda}\right)\), we have
(i) $J_\lambda(u) > 0$ for all $u \in S_r = \{u \in H^1_0(\Omega) \mid \|u\|_{H^1} = r\}$;
(ii) for any $\varepsilon > 0$ there exists a positive number $\delta \leq r$ such that

$$J_\lambda(u) \geq -\varepsilon$$

for all $u \in \{u \in H^1_0(\Omega) \mid r - \delta \leq \|u\|_{H^1} \leq r\}$.

For the positive number $r$ given in Lemma 2.1, we denote

$$B_r = \{u \in H^1_0(\Omega) \mid \|u\|_{H^1} < r\}.$$

Thus, we have the following existence results.

**Theorem 2.2** If $\tilde{\lambda}$ is chosen as in Lemma 2.1 and $\lambda \in \left(0, \tilde{\lambda}\right)$, then there is a $u_0 \in B_r$ such that

$$J_\lambda(u_0) = \inf \{J_\lambda(u) \mid u \in \overline{B_r}\} < 0,$$

and $u_0$ is a positive solution of equation $(E_\lambda)$.

Let us define

$$D_\lambda = \{0 < \lambda < \infty \mid \text{the equation (E}_\lambda\text{) has a positive solution}\};$$

$$\lambda_0 = \sup D_\lambda.$$

By Theorem 2.2, $D_\lambda$ is a nonempty set. Moreover, we have the following results.

**Lemma 2.3** For all $\lambda \in (0, \lambda_0)$, the equation $(E_\lambda)$ has a minimal solution.

**Proof.** See the proof of theorem 3.3 in Zhu-Zhou [33].

**Lemma 2.4** Let $v \in H^1_0(\Omega) \setminus \{0\}$ is nonnegative. Then

$$\mu(v) = \inf \left\{ \int_\Omega |\nabla u|^2 + u^2 \mid u \in H^1_0(\Omega), (p-1) \lambda \int_\Omega v^{p-2} u^2 = 1 \right\}$$

is achieved by some $u \geq 0$. Furthermore, if $u_\lambda$ is a minimal positive solution of equation $(E_\lambda)$, then $\mu(u_\lambda) > 1$.

**Proof.** This easy to see that $\mu(v) < \infty$. Let $\{u_n\} \subset H^1_0(\Omega)$ be a minimizing sequence of $\mu(v)$, that is

$$(p-1) \lambda \int_\Omega v^{p-2} u_n^2 = 1 \text{ and } \int_\Omega |\nabla u_n|^2 + u_n^2 = \mu(v) + o(1).$$
Clearly, \( \{ u_n \} \) is bounded in \( H^1_0(\Omega) \). Hence, there exist a subsequence \( \{ u_n \} \) and \( \overline{u} \in H^1_0(\Omega) \) such that

\[
\begin{align*}
  u_n &\rightharpoonup \overline{u} \text{ weakly in } H^1_0(\Omega), \\
  u_n &\to \overline{u} \text{ a.e. in } \Omega.
\end{align*}
\]

Moreover,

\[
\int_{\Omega} |\nabla \overline{u}|^2 + \overline{u}^2 \leq \liminf_{n \to \infty} \int_{\Omega} |\nabla u_n|^2 + u_n^2 = \mu(v).
\]

Since \( \{ u_n \} \) is bounded in \( H^1_0(\Omega) \) and \( u_n \to \overline{u} \) a.e. in \( \Omega \), by the Rellich–Kondrachov theorem and the H"older inequality we have

\[
(p - 1) \lambda \int_{\Omega} v^{p-2} (u_n - \overline{u})^2 = o(1),
\]

this implies \( (p - 1) \lambda \int_{\Omega} v^{p-2} \overline{u}^2 = 1 \). Thus, \( \overline{u} \) achieves \( \mu(v) \). Since \( |\overline{u}| \) also achieves \( \mu(v) \). Hence, we may assume \( \overline{u} \geq 0 \) in \( \Omega \) and \( \overline{u} \) satisfies

\[
-\Delta \overline{u} + \overline{u} = \mu(v) (p - 1) \lambda v^{p-2} \overline{u} \text{ in } \Omega \tag{3}
\]

By the maximum principle for weak solutions we deduce that \( \overline{u} > 0 \) in \( \Omega \). Similar to the proof of lemma 3.1 in Cao-Zhou [14], if \( u_\lambda \) is a minimal positive solution of equation \( (E_\lambda) \), then \( \mu(u_\lambda) > 1 \).

\[\square\]

**Theorem 2.5** \( 0 < \lambda_0 < \infty \).

**Proof.** If \( \lambda_0 = \infty \), then there exists a sequence \( \{ \lambda_n \} \subset (1, \infty) \) such that \( \lambda_n \not\to \infty \). By Lemmas 2.3, 2.4, for each \( n \) the equation \( (E_{\lambda_n}) \) has a positive solution \( u_n = u_{\lambda_n} \) and \( \mu(u_n) > 1 \). Thus, for each \( n \) we have

\[
\int_{\Omega} |\nabla \varphi|^2 + \varphi^2 \geq \mu(u_n) (p - 1) \lambda_n \int_{\Omega} u_n^{p-2} \varphi^2 \text{ for all } \varphi \in H^1_0(\Omega) \setminus \{0\}. \tag{4}
\]

We introduce the linear equation

\[
\begin{cases}
-\Delta v + v = h(x), \\
0 < v \in H^1_0(\Omega).
\end{cases} \tag{5}
\]

It is known that (5) has a unique positive solution \( v_0 \) and we can prove that \( u_n \geq v_0 \) for all \( n \). Indeed, we know that

\[
-\Delta u_n + u_n = \lambda_n u_n^{p-1} + \lambda_n h(x) \text{ in } \Omega, \\
-\Delta v_0 + v_0 = h(x) \text{ in } \Omega.
\]
Since \( \{\lambda_n\} \subset (1, \infty) \), we obtain
\[
-\Delta (u_n - v_0) + (u_n - v_0) = \lambda_n u_n^{p-1} + (\lambda_n - 1) h(x) > 0 \text{ in } \Omega.
\]
Then the strong maximum principle implies that \( u_n > v_0 \) in \( \Omega \). Moreover, by (4) we can conclude
\[
\int_{\Omega} |\nabla \varphi|^2 + \varphi^2 > (p - 1) \lambda_n \int_{\Omega} v_0^{p-2} \varphi^2.
\]
That is
\[
\lambda_n < \frac{\int_{\Omega} |\nabla \varphi|^2 + \varphi^2}{(p - 1) \int_{\Omega} v_0^{p-2} \varphi^2}, \text{ for all } \varphi \in H^1_0(\Omega) \setminus \{0\}.
\]
This completes the proof of theorem. \( \square \)

3 Uniqueness of Positive Solution

We now prove Theorem 1.2: (i) From Theorem 1.1 and Lemma 2.4, the equation \( (E_{\lambda}) \) has a minimal positive solution \( u_{\lambda} \) such that \( \mu(u_{\lambda}) > 1 \). Suppose, by contradiction, the equation \( (E_{\lambda}) \) has a second positive solution \( w_{\lambda} \) such that \( \mu(w_{\lambda}) > 1 \). Since \( u_{\lambda} \) is a minimal positive solution of equation \( (E_{\lambda}) \), we have \( w_{\lambda} > u_{\lambda} \). Then \( w_{\lambda} - u_{\lambda} > 0 \). Again using Lemma 2.4
\[
\int_{\Omega} |\nabla (w_{\lambda} - u_{\lambda})|^2 + (w_{\lambda} - u_{\lambda})^2 dx = \lambda \int_{\Omega} [w_{\lambda}^{p-1} - u_{\lambda}^{p-1}] (w_{\lambda} - u_{\lambda}) dx
\]
\[
\leq (p - 1) \lambda \int_{\Omega} w_{\lambda}^{p-2} (w_{\lambda} - u_{\lambda})^2 dx
\]
\[
< \mu(w_{\lambda})(p - 1) \lambda \int_{\Omega} w_{\lambda}^{p-2} (w_{\lambda} - u_{\lambda})^2 dx
\]
\[
\leq \int_{\Omega} |\nabla (w_{\lambda} - u_{\lambda})|^2 + (w_{\lambda} - u_{\lambda})^2 dx,
\]
which is a contradiction. This completes the proof of part (i)

(ii) Assume that equation \( (E_{\lambda}) \) has another positive solution \( v' \) such that \( u_{\lambda} < v < v' \). Let \( \overline{u} \) be a minimizer of \( \mu(v) \). Then by (3),
\[
\int_{\Omega} \nabla (v' - v) \nabla \overline{u} + (v' - v) \overline{u} = \lambda \int_{\Omega} [(v')^{p-1} - v^{p-1}] \overline{u}
\]
\[
\geq (p - 1) \lambda \int_{\Omega} v^{p-2} (v' - v) \overline{u}
\]
\[
> \mu(v)(p - 1) \lambda \int_{\Omega} v^{p-2} \overline{u} (v' - v)
\]
\[
= \int_{\Omega} \nabla \overline{u} \nabla (v' - v) + \overline{u} (v' - v),
\]
which is a contradiction. \( \square \)
4 Three Positive Solutions

Associated with equation (1), we consider the following minimization problem: for \( u \in H^1_0(\Omega) \) define

\[
I_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + u^2 - \frac{1}{p} \int_\Omega |u|^p - \lambda^{\frac{p-1}{p-2}} \int_\Omega hu;
\]

\[
M_\lambda = \{ u \in H^1_0(\Omega) \setminus \{0\} \mid \langle I'_\lambda(u) , u \rangle = 0 \};
\]

\[
\alpha_\lambda = \inf \{ I_\lambda(u) \mid u \in M_\lambda \}.
\]

It is well known that the solutions of equation (1) are the critical points of the energy functional \( I_\lambda \) (see Rabinowitz [26]). We define the Palais–Smale (denoted by (PS)) sequence and (PS)–condition for \( I_\lambda \) as follows.

**Definition 4.1**

(i) For \( \beta \in \mathbb{R} \), a sequence \( \{u_n\} \) is a \((PS)_\beta\)-sequence in \( H^1_0(\Omega) \) for \( I_\lambda \) if \( I_\lambda(u_n) = \beta + o(1) \) and \( I'_\lambda(u_n) = o(1) \) strongly in \( H^{-1}(\Omega) \) as \( n \to \infty \).

(ii) \( I_\lambda \) satisfies the \((PS)_\beta\)-condition if every \((PS)_\beta\)-sequence in \( H^1_0(\Omega) \) for \( I_\lambda \) contains a convergent subsequence.

Now we study the break down of the \((PS)\)-condition for \( I_\lambda \). First, we introduce the following elliptic equation in \( \mathbb{R}^N \):

\[
\begin{cases}
-\Delta u + u = u^{p-1} & \text{in } \mathbb{R}^N, \\
u > 0 & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N).
\end{cases}
\]

We define the energy functional \( I_0 : H^1(\mathbb{R}^N) \to \mathbb{R} \) as follows

\[
I_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p.
\]

Using the results of Berestycki-Lions [10] and Kwong [20], the equation (6) has a unique positive radial solution \( w(x) \) up to translation such that

\[
\alpha_0(\mathbb{R}^N) = I_0(w) = \inf \{ I_0(u) \mid \text{for any solution } u \neq 0 \text{ of equation (6)} \}
\]

and for any \( \delta > 0 \) and \( x \in \mathbb{R}^N \),

\[
w(x) \leq C \exp(- (1 - \delta) |x|) \text{ and } |\nabla w(x)| \leq C \exp(- (1 - \delta) |x|)
\]

for some \( C > 0 \). Moreover, the unique solution \( w(x) \) of equation (6) plays an important role in describing the asymptotic behavior of a \((PS)\)-sequence for \( I_\lambda \).
Proposition 4.2 Let \( \{u_n\} \) be a (PS)-sequence in \( H_0^1(\Omega) \) for \( I_\lambda \). Then there exist a subsequence \( \{u_n\} \), an integer \( m \in \mathbb{N} \cup \{0\} \), \( m \) sequences \( \{x_n^1\}, \{x_n^2\}, \ldots, \{x_n^m\} \subset \mathbb{R}^N \) and a critical point \( u_0 \in H_0^1(\Omega) \) of \( I_\lambda \) such that
\[
\begin{align*}
|x_n^i| & \to \infty \text{ for } 1 \leq i \leq m, \\
x_n^i - x_n^j & \to \infty \text{ for } 1 \leq i, j \leq m \text{ and } i \neq j, \\
u_n & \rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega), \\
u_n & = u_0 + \sum_{i=1}^m w(\cdot - x_n^i) + o(1) \text{ strongly in } H^1(\mathbb{R}^N), \\
I_\lambda(u_n) & = I_\lambda(u_0) + mI_0(w) + o(1).
\end{align*}
\]

Proof. This is a standard result. See Benci-Cerami [9] and Lions [24] for analogous arguments. \( \square \)

Corollary 4.3 \( I_\lambda \) satisfies the (PS)\( \beta \)-condition for \( \beta < \alpha_\lambda + \alpha_0 \left( \mathbb{R}^N \right) \).

Proof. Similar to the proof of corollary 1.10 in Adachi-Tanaka [1]. \( \square \)

Next, we give some properties of the functional \( I_0 \).

Lemma 4.4 We have
\[
\begin{align*}
(i) & \inf \{ I_0(u) \mid u \in M_0 \} = \alpha_0 \left( \mathbb{R}^N \right), \\
(ii) & \inf \{ I_0(u) \mid u \in M_0 \} \text{ is not achieved}, \\
(iii) & I_0 \text{ satisfies the (PS)\( \beta \)-condition for } \beta \in (\alpha_0 \left( \mathbb{R}^N \right), 2\alpha_0 \left( \mathbb{R}^N \right)) \).
\end{align*}
\]

Proof. See Benci-Cerami [9]. \( \square \)

Finally, we establish the regularity and the decay estimate for solutions of equation \((E_\lambda)\). In what follows, we denote by \( \rho_0 > 0 \) a minimum number such that \( \omega \subset B^N(0; \rho_0) \). Then we have the following results.

Lemma 4.5 If \( u \in H_0^1(\Omega) \) is a weak solution of equation \((E_\lambda)\), then
\[
\begin{align*}
(i) & u \in C(\Omega) \cap L^q(\Omega) \text{ for all } 2 \leq q < \infty \text{ and } u(x) \to 0 \text{ as } |x| \to \infty; \\
(ii) & \text{for each } \varepsilon > 0, \text{ there exist constants } C > 0 \text{ and } R_1 > \rho_0 \text{ such that } \\
u(x) & \geq C \exp \left( -(1 + \varepsilon) |x| \right), \text{ for all } |x| \geq R_1.
\end{align*}
\]

Proof. See the proof of lemma 3.5 in Zhu-Zhou [33]. \( \square \)

Remark 4.1 If \( u_0 \) is a positive solution of equation \((E_\lambda)\), then \( v_0 = \lambda^{1/(p-2)}u_0 \) is a positive solution of equation (1). Furthermore, by Lemma 4.5 we have
\[
v_0(x) \geq \lambda^{1/(p-2)}C \exp \left( -(1 + \varepsilon) |x| \right), \text{ for all } |x| \geq R_1.
\]
4.1 Existence of a local minimum

Define

\[ \psi_\lambda (u) = \langle I'_\lambda (u), u \rangle = \| u \|_{H^1}^2 - \int_\Omega |u|^p - \lambda \frac{p-1}{p-2} \int_\Omega hu. \]

Then we have the following results.

**Lemma 4.6** There exists \( \lambda_1 > 0 \) such that for each \( \lambda \in (0, \lambda_1) \) and \( u \in M_\lambda \) we have

\[ \langle \psi'_\lambda (u), u \rangle = \| u \|_{H^1}^2 - (p-1) \int_\Omega |u|^p \neq 0. \]

**Proof.** Our proof is almost the same as that in Tarantello [27]. \( \square \)

Throughout this section, let \( \lambda \in (0, \lambda_1) \). Then we write

\[ M_\lambda = M_\lambda^+ \cup M_\lambda^-, \]

where

\[ M_\lambda^+ = \left\{ u \in M_\lambda \mid \| u \|_{H^1}^2 - (p-1) \int_\Omega |u|^p > 0 \right\}, \]

\[ M_\lambda^- = \left\{ u \in M_\lambda \mid \| u \|_{H^1}^2 - (p-1) \int_\Omega |u|^p < 0 \right\}, \]

and define

\[ \alpha_\lambda^+ = \inf \{ I_\lambda (u) \mid u \in M_\lambda^+ \} ; \quad \alpha_\lambda^- = \inf \{ I_\lambda (u) \mid u \in M_\lambda^- \}. \]

For each \( u \in H^1_0 (\Omega) \setminus \{0\} \), we define

\[ t_{\text{max}} = \left( \frac{\| u \|_{H^1}^2}{(p-1) \int_\Omega |u|^p} \right)^{\frac{1}{p-2}} > 0. \]

Then we have the following lemma.

**Lemma 4.7** For each \( u \in H^1_0 (\Omega) \setminus \{0\} \), we have

(i) there is a unique \( t^- = t^- (u) > t_{\text{max}} > 0 \) such that \( t^- u \in M_\lambda^- \) and \( I_\lambda (t^- u) = \max_{t \geq t_{\text{max}}} I_\lambda (tu) \);

(ii) \( t^- (u) \) is a continuous function for nonzero \( u \);

(iii) \( M_\lambda^- = \left\{ u \in H^1_0 (\Omega) \setminus \{0\} \mid \frac{1}{\| u \|_{H^1}} t^- \left( \frac{u}{\| u \|_{H^1}} \right) = 1 \right\} \);

(iv) if \( \int_\Omega h u > 0 \), then there is a unique \( 0 < t^+ = t^+ (u) < t_{\text{max}} \) such that \( t^+ u \in M_\lambda^+ \) and \( I_\lambda (t^+ u) = \min_{0 \leq t \leq t^-} I_\lambda (tu) \).

**Proof.** Similar to the proof of some results in Tarantello [27]. \( \square \)
For $c > 0$, we define
\[
I_{0,c}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + u^2 - \frac{1}{p} \int_{\Omega} c |u|^p ;
\]
\[
M_{0,c} = \{ u \in H^1_0(\Omega) \setminus \{0\} \mid \langle I_{0,c}(u), u \rangle = 0 \} ;
\]
\[
M_0 = \{ u \in H^1_0(\Omega) \setminus \{0\} \mid \langle I_0(u), u \rangle = 0 \} .
\]

Note that $I_0 = I_{0,c}$ for $c = 1$, and for each $u \in M_0^\mu$ there is a unique $t^1 = t^1(u) > 0$ such that $t^1u \in M_0$. Then we have the following results.

**Lemma 4.8** For each $u \in M_0^\mu$, we have
(i) there is a unique $t^c(u) > 0$ such that $t^c(u)u \in M_{0,c}$ and
\[
\max_{t \geq 0} I_{0,c}(tu) = I_{0,c}(t^c(u)u) = \left(\frac{1}{2} - \frac{1}{p}\right) c^{\frac{p-2}{2}} \left[ \frac{|u|_{H^1}}{|u|_{L^p}} \right]^{\frac{2p}{p-2}} ;
\]
(ii) for $\mu \in (0, 1)$,
\[
I_\mu(u) \geq \left(1 - \lambda_{p-2}^{\frac{p-1}{p-2}} \mu \right)^\frac{p}{p-2} I_0(t^1u) - \frac{\lambda_{p-2}^{\frac{p-1}{p-2}}}{2\mu} \|h\|^2_{L^2} .
\]

**Proof.** (i) Similar to the proof of some result in Brown-Zhang [12].
(ii) For each $u \in M_0^\mu$, let $c = 1/ \left(1 - \lambda_{p-2}^{\frac{p-1}{p-2}} \mu \right)$, $t^c = t^c(u) > 0$ and $t^1 = t^1(u) > 0$ such that $t^c u \in M_{0,c}$ and $t^1 u \in M_0$. For $\mu \in (0, 1)$, we have
\[
\int_{\Omega} h t^c u dx \leq \|t^c u\|_{H^1} \|h\|_{L^2} \leq \frac{\mu}{2} \|t^c u\|^2_{H^1} + \frac{1}{2\mu} \|h\|^2_{L^2} .
\]

Then by part (i),
\[
\sup_{t \geq 0} I_\mu(tu) \geq I_\mu(t^c u) \geq \frac{1}{2c} \|t^c u\|^2_{H^1} - \frac{1}{p} \int_{\Omega} |t^c u|^p - \frac{\lambda_{p-2}^{\frac{p-1}{p-2}}}{2\mu} \|h\|^2_{L^2} = \frac{1}{c} \left[ \frac{1}{2} \|t^c u\|^2_{H^1} - \frac{1}{p} \int_{\Omega} c |t^c u|^p \right] - \frac{\lambda_{p-2}^{\frac{p-1}{p-2}}}{2\mu} \|h\|^2_{L^2} = \frac{1}{c} I_{0,c}(t^c u) - \frac{\lambda_{p-2}^{\frac{p-1}{p-2}}}{2\mu} \|h\|^2_{L^2}
\]
\[
= \left(1 - \lambda_{p-2}^{\frac{p-1}{p-2}} \mu \right)^\frac{p}{p-2} \left(\frac{1}{2} - \frac{1}{p}\right) \left[ \frac{|u|_{H^1}}{|u|_{L^p}} \right]^{\frac{2p}{p-2}} - \frac{\lambda_{p-2}^{\frac{p-1}{p-2}}}{2\mu} \|h\|^2_{L^2} = \left(1 - \lambda_{p-2}^{\frac{p-1}{p-2}} \mu \right)^\frac{p}{p-2} I_0(t^1 u) - \frac{\lambda_{p-2}^{\frac{p-1}{p-2}}}{2\mu} \|h\|^2_{L^2} .
\]
By Lemma 4.7 (i), \[ \sup_{t \geq 0} I_\lambda (tu) = I_\lambda (u) . \]

Thus, \[ I_\lambda (u) \geq \left( 1 - \lambda^{\frac{p-1}{2}} \mu \right)^{\frac{p}{p-2}} I_0 (t^1 u) - \frac{\lambda^{\frac{p-1}{p-2}}}{2 \mu} \| h \|_{L^2}^2 . \]

This completes the proof. \qed

Lemma 4.9 (i) For each \( u \in M^+_\lambda \), we have \( \int_\Omega h u > 0 \) and \( I_\lambda (u) < 0 \). In particular, \( \alpha_\lambda \leq \alpha^+_\lambda < 0 \).

(ii) \( I_\lambda \) is coercive and bounded below on \( M_\lambda \).

Proof. Similar to the proof of theorem 1 in Tarantello [27]. \qed

Furthermore, we have the following result.

Theorem 4.10 For each \( \lambda \in (0, \lambda_1) \), the equation (1) has a positive solution \( v_\lambda \in M^+_\lambda \) such that \( I_\lambda (v_\lambda) = \alpha_\lambda = \alpha^+_\lambda \). Furthermore, \( \| v_\lambda \|_{H^1} \to 0 \) as \( \lambda \to 0 \).

Proof. Similar to the proof of theorem 2.1 in Cao-Zhu [14]. \qed

4.2 Existence of Two Solutions

In this subsection, we prove that the equation (\( E_\lambda \)) has at least two positive solutions. Let \( w(x) \) be a unique positive radial solution of equation (6) and \( e \in S^{N-1} = \{ x \in \mathbb{R}^N \mid |x| = 1 \} \). We denote \[ w_l (x) = w(x + le), \ l \in (0, \infty). \]

Recall that \( \Omega = \mathbb{R}^N \setminus \omega \), where \( \omega \subset B^N (0, \rho_0) \) and let \( \psi : \mathbb{R}^N \to [0, 1] \) be a \( C^\infty \) function defined by \( \psi (x) = \overline{\psi} (\frac{|x|}{\rho_0}) \), where \( \overline{\psi} : \mathbb{R}^+ \cup \{ 0 \} \to [0, 1] \) is a \( C^\infty \) nondecreasing function such that \[ \overline{\psi} (t) = \begin{cases} 0, & t \leq 1; \\ 1, & t \geq 2. \end{cases} \]

Clearly, \( \psi w_l \in H^1_0 (\Omega) \). Then we have the following results.

Lemma 4.11 (i) \( \lim_{l \to \infty} \| \psi w_l (x) - w_l (x) \|_{H^1} = 0 \) uniformly in \( e \in S^{N-1} \).

(ii) \( \lim_{l \to \infty} I_0 (\psi w_l) = \alpha_0 (\mathbb{R}^N) \) uniformly in \( e \in S^{N-1} \).

Proof. Similar to the proof of theorem 2.4 in Benci-Cerami [9]. \qed
Lemma 4.12 (i) For each $\lambda \in (0, \lambda_1)$ there exists $t_0 > 0$ such that for $t > 2\rho_0 + 1$,
\[ I_{\lambda} (v_\lambda + t\psi w_l) < I_{\lambda} (v_\lambda) \quad \text{for all } t \geq t_0 \text{ and } e \in S^{N-1}. \]

(ii) For each $\lambda \in (0, \lambda_1)$ there exists $l_1 > 0$ such that for $l > l_1$,
\[ \sup_{t \geq 0} I_{\lambda} (v_\lambda + t\psi w_l) < I_{\lambda} (v_\lambda) + \alpha_0 (\mathbb{R}^N) = \alpha_\lambda + \alpha_0 (\mathbb{R}^N), \]
where $v_\lambda$ is the local minimum in Theorem 4.10.

**Proof.** (i) By the definition of $\psi$ and $v_\lambda$ is a positive solution of equation $(E_\lambda)$, using the fact that $\int_\Omega \nabla v_\lambda \nabla \psi w_l = -\int \Delta v_\lambda \psi w_l$ and $\psi (x) \equiv 1$ whenever $|x| > 2\rho_0$, we have
\[
I_{\lambda} (v_\lambda + t\psi w_l) = I_{\lambda} (v_\lambda) + \frac{t^2}{2} \int_\Omega |\nabla \psi w_l|^2 + \psi^2 w_l^2 dx - \int_\Omega \int_0^{t\psi w_l} [(v_\lambda + s)^{p-1} - v_\lambda^{p-1}] ds dx \]
\[
\leq I_{\lambda} (v_\lambda) + \frac{t^2}{2} \int_\Omega |\nabla \psi w_l|^2 + \psi^2 w_l^2 dx + t \int_\Omega w_l v_\lambda^{p-1} dx - \frac{1}{p} \int_{B^{N(0;2\rho_0)^c}} (tw_l)^p dx \]
\[
\leq I_{\lambda} (v_\lambda) + \frac{t^2}{2} \int_\Omega |\nabla \psi w_l|^2 + \psi^2 w_l^2 dx + t \int_\Omega w_l v_\lambda^{p-1} dx - \frac{tp}{p} \int_{B^{N(l;1)}} w_l^p dx \]
\[
= I_{\lambda} (v_\lambda) + \frac{t^2}{2} \int_\Omega |\nabla \psi w_l|^2 + \psi^2 w_l^2 dx + t \int_\Omega w_l v_\lambda^{p-1} dx - \frac{tp}{p} \int_{B^{N(0;1)}} w_l^p dx. \]

Since $p > 2$, by Lemma 4.11 we can choose $t_0 > 0$ large enough such that (i) holds.

(ii) Since $I_{\lambda}$ is continuous in $H^1_0 (\Omega)$, there exists $t_1 > 0$ such that for $l > 2\rho_0 + 1$,
\[ I_{\lambda} (v_\lambda + t\psi w_l) < I_{\lambda} (v_\lambda) + \alpha_0 (\mathbb{R}^N) \quad \text{for all } t < t_1 \text{ and } e \in S^{N-1}. \]

Using part (i) we know that for $l > 2\rho_0 + 1$,
\[ \sup_{t \geq t_0} I_{\lambda} (v_\lambda + t\psi w_l) < I_{\lambda} (v_\lambda) + \alpha_0 (\mathbb{R}^N) \quad \text{for all } e \in S^{N-1}. \]

Thus, we only need to show that there exists $l_0 > 0$ such that for $l > l_0$,
\[ \sup_{t_1 \leq t \leq t_0} I_{\lambda} (v_\lambda + t\psi w_l) < I_{\lambda} (v_\lambda) + \alpha_0 (\mathbb{R}^N) \quad \text{for all } e \in S^{N-1}. \]

By Brown-Zhang [12] and Willem [29], we know that
\[ \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla w_l|^2 + w_l^2 dx - \frac{tp}{p} \int_{\mathbb{R}^N} w_l^p dx \leq \alpha_0 (\mathbb{R}^N). \]
For \( l > 2\rho_0 + 1 \) and \( t_1 \leq t \leq t_0 \),
\[
I_\lambda (v_\lambda + t\psi w_l) = I_\lambda (v_\lambda) + \frac{t^2}{2} \int_\Omega |\nabla \psi w_l|^2 + \psi^2 w_l^2 dx - \int_\Omega \int_0^{t\psi w_l} [(v_\lambda + s)^{p-1} - v_\lambda^{p-1}] ds dx
\]
\[
\leq I_\lambda (v_\lambda) + \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla \psi|^2 (|\nabla w_l|^2 + w_l^2) dx + \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla \psi|^2 w_l^2 dx
\]
\[
+ t^2 \int_{\mathbb{R}^N} |\nabla \psi| |\nabla w_l| |w_l| dx - \int_{\mathbb{R}^N} \int_0^{t\psi w_l} (v_\lambda + s)^{p-1} - v_\lambda^{p-1} ds dx
\]
\[
\leq I_\lambda (v_\lambda) + \alpha_0 (\mathbb{R}^N) + \frac{t^2}{2} \int_{\text{supp}(\nabla \psi)} |\nabla \psi|^2 w_l^2 + 2 |\nabla \psi| |\nabla w_l| |w_l| dx
\]
\[
+ \int_{B^{(0;2\rho_0)}} \int_0^{t_1 w_l} s^{p-1} ds dx - \int_{B^{(le;1)}} \int_0^{t_1 w_l} (v_\lambda + s)^{p-1} - v_\lambda^{p-1} - s^{p-1} ds dx
\]
\[
\leq I_\lambda (v_\lambda) + \alpha_0 (\mathbb{R}^N) + \frac{t_0^2}{2} \int_{\text{supp}(\nabla \rho)} |\nabla \psi|^2 w_l^2 + 2 |\nabla \psi| |\nabla w_l| |w_l| dx
\]
\[
+C \int_{B^{(0;2\rho_0)}} (t_0 w_l)^2 + (t_0 w_l)^p dx - \int_{B^{(le;1)}} \int_0^{t_1 w_l} (v_\lambda + s)^{p-1} - v_\lambda^{p-1} - s^{p-1} ds dx.
\] (9)

By the Taylor expansion
\[
\int_0^{t_1 w_l} (v_\lambda + s)^{p-1} - v_\lambda^{p-1} - s^{p-1} ds \geq \int_0^{t_1 w_l} (p - 1) s^{p-2} v_\lambda - v_\lambda^{p-1} ds
\]
\[
= [(t_1 w_l)^{p-2} - v_\lambda^{p-2}] t_1 w_l v_\lambda.
\] (10)

Since \( w_l > 0 \) in \( \mathbb{R}^N \), there exists a number \( c_1 > 0 \) such that
\[
w_l \geq c_1 \text{ in } B^N (le; 1)
\] (11)

Since \( v_\lambda (x) \rightarrow 0 \) as \( |x| \rightarrow \infty \), it follows from the definition of \( w_l \) that for \( l \) large enough,
\[
t_1 w_l > v_\lambda \text{ in } B^N (le; 1).
\]
Thus, there exist $c_2 > 0$ and $l_0 > 0$ such that for $l > l_0$,
\[(t_1 w_l)^{p-2} - v_\lambda^{p-2} > c_2. \quad (12)\]

Then combining (10) – (12) we have
\[
\int_{B_{(l\epsilon;1)}} \int_0^{t_1 w_l} (v_\lambda + s)^{p-1} - v_\lambda^{p-1} - s^{p-1} ds \geq c_1 c_2 \int_{B_{(l\epsilon;1)}} t_1 v_\lambda dx.
\]

By (8), we find that for any $\varepsilon > 0$, there exists $C_1 > 0$ such that
\[
\int_{B_{(l\epsilon;1)}} \int_0^{t_1 w_l} (v_\lambda + s)^{p-1} - v_\lambda^{p-1} - s^{p-1} ds \geq C_1 \lambda^{-\frac{1}{p-2}} \exp((-1 + \varepsilon) l), \quad (13)
\]
for all $l \geq \max \{l_0, R_1 + 1\}$, where $R_1$ is given in Lemma 4.5 (ii). It follows from (7) that for any $\delta > 0$, there exists a constant $C_2 > 0$ such that for $l \geq 0$,
\[
\frac{t_0^2}{2} \int_{\text{supp}(\nabla \psi)} |\nabla \psi|^2 w_l^2 + 2 |\nabla \psi| |\nabla w_l| |w_l| \, dx \leq C_2 \exp(-2(1 - \delta) l) \quad (14)
\]
and
\[
\int_{B_{(0;2\rho_0)}} (t_0 w_l)^2 + (t_0 w_l)^p \, dx \leq C_2 \exp(-2(1 - \delta) l) \quad (15)
\]
Then from (9) and using (13) – (15), we find that
\[
I_\lambda (v_\lambda + t\psi w_l)
\leq I_\lambda (v_\lambda) + \alpha_0 \left( \mathbb{R}^N \right) + C_2 \exp(-2(1 - \delta) l) - C_1 \lambda^{-\frac{1}{p-2}} \exp((-1 + \varepsilon) l)
\]
for all $t \in [t_1, t_0]$ and $l \geq \max \{l_0, R_1 + 1\}$. Choosing positive numbers $\delta, \varepsilon$ such that $2\delta + \varepsilon < 1$, we can find some $l_1 > \max \{l_0, R_1 + 1\}$ large enough such that
\[
C_2 \exp(-2(1 - \delta) l) - C_1 \lambda^{-\frac{1}{p-2}} \exp((-1 + \varepsilon) l) < 0 \text{ for all } l \geq l_1. \quad (16)
\]
This completes the proof of (ii).

\[\Box\]

**Remark 4.2** By inequality (16), we have $l_1 \to \infty$ as $\lambda \to 0$.

In the following, we use an idea of Adachi-Tanaka [1]. For $c \in \mathbb{R}$, we denote
\[
[I_\lambda \leq c] = \{u \in M_\lambda^- : I_\lambda (u) \leq c\}.
\]

We then try to show for a sufficiently small $\sigma > 0$
\[
\text{cat} \left( [I_\lambda \leq \alpha_\lambda + \alpha_0 \left( \mathbb{R}^N \right) - \sigma] \right) \geq 2. \quad (17)
\]
To prove (17), we need some preliminaries. Recall the definition of Lusternik-Schnirelman category.
Definition 4.13 (i) For a topological space $X$, we say a non-empty, closed subset $Y \subset X$ is contractible to a point in $X$ if and only if there exists a continuous mapping 

$$\eta : [0, 1] \times Y \to X$$

such that for some $x_0 \in X$

$$\eta(0, x) = x \text{ for all } x \in Y,$$

and

$$\eta(1, x) = x_0 \text{ for all } x \in Y.$$

(ii) We define

$$\text{cat} (X) = \min \{ k \in \mathbb{N} \mid \text{there exist closed subsets } Y_1, ..., Y_k \subset X \text{ such that } Y_j \text{ is contractible to a point in } X \text{ for all } j \text{ and } \bigcup_{j=1}^{k} Y_j = X \}.$$

When there do not exist finitely many closed subsets $Y_1, ..., Y_k \subset X$ such that $Y_j$ is contractible to a point in $X$ for all $j$ and $\bigcup_{j=1}^{k} Y_j = X$, we say $\text{cat} (X) = \infty$.

We need the following two lemmas.

Lemma 4.14 Suppose that $X$ is a Hilbert manifold and $\Psi \in C^1 (X, \mathbb{R}).$ Assume that there are $c_0 \in \mathbb{R}$ and $k \in \mathbb{N},$

(i) $\Psi (x)$ satisfies the Palais–Smale condition for energy level $c \leq c_0$;

(ii) $\text{cat} \left( \{ x \in X \mid \Psi (x) \leq c_0 \} \right) \geq k.$

Then $\Psi (x)$ has at least $k$ critical points in $\{ x \in X ; \Psi (x) \leq c_0 \}$.

Proof. See Ambrosetti [4, Theorem 2.3].

Lemma 4.15 Let $N \geq 1$, $S^{N-1} = \{ x \in \mathbb{R}^N ; |x| = 1 \}$, and let $X$ be a topological space. Suppose that there are two continuous maps

$$F : S^{N-1} \to X, \ G : X \to S^{N-1}$$

such that $G \circ F$ is homotopic to the identity map of $S^{N-1}$, that is, there exists a continuous map $\zeta : [0, 1] \times S^{N-1} \to S^{N-1}$ such that

$$\zeta (0, x) = (G \circ F) (x) \text{ for each } x \in S^{N-1},$$

$$\zeta (1, x) = x \text{ for each } x \in S^{N-1}.$$

Then

$$\text{cat} (X) \geq 2.$$
Proof. See Adachi-Tanaka [1, Lemma 2.5].

Let

\[ A_1 = \left\{ u \in H^1_0 (\Omega) \setminus \{0\} \mid \frac{1}{\| u \|_{H^1}} t^- \left( \frac{u}{\| u \|_{H^1}} \right) > 1 \right\} \cup \{0\}; \]
\[ A_2 = \left\{ u \in H^1_0 (\Omega) \setminus \{0\} \mid \frac{1}{\| u \|_{H^1}} t^- \left( \frac{u}{\| u \|_{H^1}} \right) < 1 \right\}. \]

Follows from Tarantello [27], we have the following results.

Lemma 4.16

(i) \( H^1_0 (\Omega) \setminus M_\lambda^- = A_1 \cup A_2. \)

(ii) \( M_\lambda^+ \subset A_1. \)

(iii) For each \( \lambda \in (0, \lambda_1) \) there exist \( t_\ast > 1 \) and \( l_2 \geq l_1 \) such that \( v_\lambda + t_\ast \psi w_1 \in A_2 \) for all \( l \geq l_2, \) where \( l_1 \) is defined as in Lemma 4.12.

(iv) For each \( l \geq l_2 \) there exists \( s_l \in (0, 1) \) such that \( v_\lambda + s_l t_\ast \psi w_1 \in M_\lambda^- \) and

\[ s_l t_\ast = 1 + o(1) \quad \text{as} \quad \lambda \to 0. \]

(v) \( \alpha^-_\lambda < \alpha_\lambda + \alpha_0 (\mathbb{R}^N). \)

Proof. (i) By Lemma 4.7 (iii).

(ii) For each \( u \in M_\lambda^+ \), we have

\[ 1 < t_{\max} (u) < t^- (u) = \frac{1}{\| u \|_{H^1}} t^- \left( \frac{u}{\| u \|_{H^1}} \right), \]

and so \( M_\lambda^+ \subset A_1. \) In particular, \( v_\lambda \in A_1. \)

(iii) There is a constant \( c > 0 \) such that \( 0 < t^- \left( \frac{v_\lambda + t \psi w_1}{\| v_\lambda + t \psi w_1 \|_{H^1}} \right) < c \) for all \( t \geq 0 \) and \( l > 0. \) Otherwise, there exist sequences \( \{ t_n \} \) and \( \{ l_n \} \) such that \( l_n \to \infty \) as \( n \to \infty \) and \( t^- \left( \frac{v_\lambda + t_n \psi w_1}{\| v_\lambda + t_n \psi w_1 \|_{H^1}} \right) \to \infty \) as \( n \to \infty. \) Let \( w_n = v_\lambda + t_n \psi w_1. \) We claim that \( \| w_n \|_{L^p} \) is bounded below away from zero.

Case (a): \( t_n \) is bounded away from zero. Since \( w \) is a ground state solution of equation (6), we have

\[ \| \psi w_n \|_{H^1}^2 = \| \psi w_n \|_{L^p}^p + o(1) = \frac{2p}{p - 2} \alpha_0 (\mathbb{R}^N) + o (1). \]

Thus,

\[ \| w_n \|_{L^p}^p = \frac{1}{\| v_{t_n} \psi w_n \|_{H^1}^p} \int_{\Omega} \left( \frac{v_{t_n} \psi w_n}{t_n} \right)^p \]
\[ \geq \frac{\| \psi w_n \|_{L^p}^p}{2^{p-1} \left( \| v_{t_n} \|_{H^1}^p + \| \psi w_n \|_{H^1}^p \right)} \]
\[ = \frac{\alpha_0 (\mathbb{R}^N)}{2^{p-1} \left( \frac{\| v_{t_n} \|_{H^1}^p}{c_0} + \left( \frac{2p}{p - 2} \alpha_0 (\mathbb{R}^N) \right)^{\frac{p}{2}} \right)} + o(1). \]
Case (b) : there is a subsequence \( \{t_n\} \) such that \( t_n = o(1) \) as \( n \to \infty \). Then
\[
\| v_\lambda + t_n \psi w_n \|_{H^1}^2 = \| v_\lambda \|_{H^1}^2 + t_n^2 \| \psi w_n \|_{H^1}^2 + 2t_n \langle \psi w_n, v_\lambda \rangle_{H^1} = \| v_\lambda \|_{H^1}^2 + o(1).
\]
Thus,
\[
\| w_n \|_{L^p}^p \geq \frac{1}{\| v_\lambda + t_n \psi w_n \|_{H^1}^p} \int_\Omega v_\lambda^p = \frac{1}{\| v_\lambda \|_{H^1}^p} \int_\Omega v_\lambda^p + o(1).
\]
Since \( t^- (w_n) \in M^-_\lambda \subset M_\lambda \), we have
\[
I_\lambda (t^- (w_n) w_n) = \frac{1}{2} [t^- (w_n)]^2 - \frac{1}{p} [t^- (w_n)]^p \int_\Omega w_n^p - t^- (w_n) \int_\Omega h w_n \to -\infty \text{ as } n \to \infty.
\]
However, \( I_\lambda \) is bounded below on \( M_\lambda \), which is a contradiction.

Let
\[
t_* = \left( \frac{p - 2}{2p\alpha_0 (\mathbb{R}^N)} \left| c^2 - \| v_\lambda \|_{H^1}^2 \right| \right)^{\frac{1}{2}} + 1.
\]
Then
\[
\| v_\lambda + t_* \psi w_l \|_{H^1}^2 = \| v_\lambda \|_{H^1}^2 + t_*^2 \left( \frac{2p}{p - 2} \right) \alpha_0 (\mathbb{R}^N) + o(1)
\]
\[
> c^2 + o(1) \geq \left( t^- \left( \frac{v_\lambda + t_* \psi w_l}{\| v_\lambda + t_* \psi w_l \|_{H^1}} \right) \right)^2 + o(1).
\]
Thus, there exists \( l_2 \geq l_1 \) such that for \( l \geq l_2 \),
\[
\frac{1}{\| v_\lambda + t_* \psi w_l \|_{H^1}} t^- \left( \frac{v_\lambda + t_* \psi w_l}{\| v_\lambda + t_* \psi w_l \|_{H^1}} \right) < 1
\]
or \( v_\lambda + t_* \psi w_l \in A_2 \).

(iv) For each \( l \geq l_2 \), define a path \( \gamma_l (s) = v_\lambda + st_* \psi w_l \) for \( s \in [0, 1] \), then
\[
\gamma_l (0) = v_\lambda \in A_1, \; \gamma_l (1) = v_\lambda + t_* \psi w_l \in A_2.
\]

Since \( \frac{1}{\| u \|_{H^1}} t^- \left( \frac{u}{\| u \|_{H^1}} \right) \) is a continuous function for nonzero \( u \) and \( \gamma_l ([0, 1]) \) is connected, there exists \( s_l \in (0, 1) \) such that \( v_\lambda + s_l t_* \psi w_l \in M^-_\lambda \). By Lemma 4.12
\[
I_\lambda (v_\lambda + s_l t_* \psi w_l) < \alpha_\lambda + \alpha_0 (\mathbb{R}^N) < \alpha_0 (\mathbb{R}^N),
\]
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and so by Lemma 4.9 we have \( v_\lambda + s_{l_2} \psi w_l \) is uniformly bounded in \( H_0^1(\Omega) \) for all \( \lambda \in (0, \lambda_1) \) and \( l \geq l_2 \). Moreover, \( \lim_{\lambda \to 0} \|v_\lambda(x)\|_{H^1} = 0 \) and

\[
\|v_\lambda + s_{l_2} \psi w_l\|_{H^1}^2 = \int_\Omega |v_\lambda + s_{l_2} \psi w_l|^p + \lambda^{\frac{p-1}{2}} \int_\Omega h(v_\lambda + s_{l_2} \psi w_l)
\]

Thus,

\[
\|s_{l_2} \psi w_l\|_{H^1}^2 = \int_\Omega |s_{l_2} \psi w_l|^p + o(1) \text{ as } \lambda \to 0
\]

or

\[
(s_{l_2})^{p-2} = \frac{\|\psi w_l\|_{H^1}^2}{\int_\Omega |\psi w_l|^p} + o(1) \text{ as } \lambda \to 0.
\]

By Lemma 4.11 and Remark 4.2 we obtain

\[
s_{l_2} = 1 + o(1) \text{ as } \lambda \to 0.
\]

(v) By part (iv) and Lemma 4.12.

For \( l \geq l_2 \), we define a map \( F_l : S^{N-1} \to H_0^1(\Omega) \) by

\[
F_l(e)(x) = v_\lambda(x) + s_{l_2} \psi w_l(x) \text{ for } e \in S^{N-1}.
\]

Then we have the following result.

**Lemma 4.17** There exists a sequence \( \{\sigma_l\} \subset \mathbb{R}^+ \) such that

\[
F_l \left( S^{N-1} \right) \subset \left[ I_\lambda \leq \alpha_\lambda + \alpha_0 \left( \mathbb{R}^N \right) - \sigma_l \right].
\]

**Proof.** By Lemma 4.12 and Lemma 4.16 (iv), for each \( l \geq l_2 \) we have \( v_\lambda + s_{l_2} \psi w_l \in M_\lambda^{-} \) and

\[
\sup_{t \geq 0} \int_\Omega I_\lambda(v_\lambda + t \psi w_l) < \alpha_\lambda + \alpha_0 \left( \mathbb{R}^N \right) \text{ uniformly in } e \in S^{N-1}.
\]

Since \( F_l \left( S^{N-1} \right) \) is compact. Thus, \( I_\lambda(v_\lambda + s_{l_2} \psi w_l) \leq \alpha_\lambda + \alpha_0 \left( \mathbb{R}^N \right) - \sigma_l \), so that the conclusion holds.

The following lemma is a key lemma to prove our main result.

**Lemma 4.18** There exists \( \delta_0 > 0 \) such that if \( u \in M_0 \) and \( I_0(u) \leq \alpha_0 \left( \mathbb{R}^N \right) + \delta_0 \), then

\[
\int_{\mathbb{R}^N} \frac{x}{|x|} \left( |\nabla u|^2 + u^2 \right) \, dx \neq 0.
\]
Proof. On the contrary, there exists sequence \( \{ u_n \} \) in \( M_0 \) such that \( I_0 (u_n) = \alpha_0 (\mathbb{R}^N) + o(1) \) and
\[
\int_{\mathbb{R}^N} \frac{x}{|x|} \left( |\nabla u_n|^2 + u_n^2 \right) dx = 0.
\]
By Wang-Wu [28, Lemma 7], \( \{ u_n \} \) is a \((\text{PS})_{\alpha_0(\mathbb{R}^N)}\)-sequence in \( H^1_0(\Omega) \) for \( I_0 \). It follows from Proposition 4.2 and Lemma 4.4 that there exist a subsequence \( \{ u_n \} \) and a sequence \( \{ x_n \} \subset \mathbb{R}^N \) such that
\[
\begin{align*}
u_n &\rightharpoonup 0 \text{ weakly in } H^1_0(\Omega), \\
|x_n| &\to \infty \text{ as } n \to \infty \text{ and } \\
u_n (x) &= w(x - x_n) + o(1) \text{ strongly in } H^1(\mathbb{R}^N).
\end{align*}
\]
Assume \( \frac{x_n}{|x_n|} \to e \) as \( n \to \infty \), where \( e \in S^{N-1} \). Then by the Lebesgue dominated theorem, we have
\[
0 = \int_{\mathbb{R}^N} \frac{x}{|x|} \left( |\nabla u_n|^2 + u_n^2 \right) dx
= \int_{\mathbb{R}^N} \frac{x + x_n}{|x + x_n|} \left( |\nabla w|^2 + w^2 \right) dx + o(1)
= \left( \frac{2p}{p - 2} \right) e \alpha_0 (\mathbb{R}^N) + o(1),
\]
which is a contradiction. \( \square \)

Lemma 4.19 There exists \( \lambda_2 \in (0, \lambda_1) \) such that for \( \lambda \in (0, \lambda_2) \), we have
\[
\int_{\mathbb{R}^N} \frac{x}{|x|} \left( |\nabla u|^2 + u^2 \right) dx \neq 0
\]
for all \( u \in [I_\lambda < \alpha_\lambda + \alpha_0 (\mathbb{R}^N)] \).

Proof. For \( u \in [I_\lambda < \alpha_\lambda + \alpha_0 (\mathbb{R}^N)] \), there exists \( t^1 > 0 \) such that \( t^1 u \in M_0 \).
By Lemma 4.8 \( (ii) \), we have for any \( \mu \in (0, 1) \)
\[
I_0 (t^1 u) \leq \left( 1 - \lambda \frac{p+4}{p-2} \mu \right)^{-\frac{p}{p-2}} \left( I_\lambda (u) + \frac{\lambda \frac{p-1}{p-2} \|h\|^2_{L^2}}{2\mu} \right).
\]
(18)

Since \( \alpha_\lambda < 0 \), we have \([I_\lambda < \alpha_\lambda + \alpha_0 (\mathbb{R}^N)] \subset [I_\lambda < \alpha_0 (\mathbb{R}^N)] \). Thus, by (18)
\[
I_0 (t^1 u) \leq \left( 1 - \lambda \frac{p+4}{p-2} \mu \right)^{-\frac{p}{p-2}} \left( \alpha_0 (\mathbb{R}^N) + \frac{\lambda \frac{p-1}{p-2} \|h\|^2_{L^2}}{2\mu} \right).
\]
For each \( \delta_0 > 0 \) there exist \( \mu_0 > 0 \) and \( \lambda_2 \in (0, \lambda_1) \) such that for \( \lambda \in (0, \lambda_2) \),
\[
I_0 (t^1 u) < \alpha_0 (\mathbb{R}^N) + \delta_0.
\]
(19)
Since \( t^1 u \in M_0 \) and \( t^1 > 0 \), by Lemma 4.18 and (19)

\[
\int_{\mathbb{R}^N} \frac{x}{|x|} \left( |\nabla (t^1 u)|^2 + (t^1 u)^2 \right) \, dx \neq 0.
\]

This implies,

\[
\int_{\mathbb{R}^N} \frac{x}{|x|} (|\nabla u|^2 + (u)^2) \, dx \neq 0
\]

for all \( u \in [I_{\lambda} < \alpha_{\lambda} + \alpha_0 (\mathbb{R}^N)] \). \( \square \)

From Lemma 4.19, we define \( G : [I_{\lambda} < \alpha_{\lambda} + \alpha_0 (\mathbb{R}^N)] \rightarrow S^{N-1} \) by

\[
G(u) = \int_{\mathbb{R}^N} \frac{x}{|x|} (|\nabla u|^2 + |u|^2) \, dx / \left| \int_{\mathbb{R}^N} \frac{x}{|x|} (|\nabla u|^2 + |u|^2) \, dx \right|.
\]

Then we have the following results.

**Lemma 4.20** There exist \( \lambda_* \in (0, \lambda_2) \) and \( l_* \in (l_2, \infty) \) such that for \( \lambda \in (0, \lambda_*) \) and \( l \in (l_*, \infty) \), the map

\[
G \circ F_l : S^{N-1} \rightarrow S^{N-1}
\]

is homotopic to the identity.

**Proof.** Let \( \Theta = \left\{ u \in H^1_0 (\Omega) \setminus \{0\} \mid \int_{\mathbb{R}^N} \frac{x}{|x|} (|\nabla u|^2 + |u|^2) \, dx \neq 0 \right\} \). We define

\[
\overline{G} : \Theta \rightarrow S^{N-1}
\]

by

\[
\overline{G}(u) = \int_{\mathbb{R}^N} \frac{x}{|x|} (|\nabla u|^2 + |u|^2) \, dx / \left| \int_{\mathbb{R}^N} \frac{x}{|x|} (|\nabla u|^2 + |u|^2) \, dx \right|.
\]

as an extension of \( G \). By Lemma 4.16 (iv), for \( \theta \in [0, 1/2) \)

\[
(1 - 2\theta) F_l (e) + 2\theta \psi w (x + le) = \psi w (x + le) + o (1) \text{ in } H^1_0 (\Omega) \text{ as } \lambda \to 0.
\]

By an argument similar to that in Lemma 4.18, there exist \( \lambda_* \in (0, \lambda_2) \) and \( l_* \in (l_2, \infty) \) such that for \( \lambda \in (0, \lambda_*) \) and \( l \in (l_*, \infty) \),

\[
(1 - 2\theta) F_l (e) + 2\theta \psi w (x + le) \in \Theta \text{ for all } e \in S^{N-1} \text{ and } \theta \in [0, 1/2)
\]

and

\[
\psi w \left( x + \frac{l}{2 (1 - \theta)} e \right) \in \Theta \text{ for all } e \in S^{N-1} \text{ and } \theta \in [1/2, 1).
\]
We define

$$\zeta_t (\theta, e) : [0, 1] \times S^{N-1} \to S^{N-1}$$

by

$$\zeta_t (\theta, e) = \begin{cases} \overline{G} ((1-2\theta) F_t (e) + 2\theta \psi w (x + le)) & \text{for } \theta \in [0, 1/2); \\ \overline{G} \left( \psi w \left( x + \frac{l}{2(1-\theta)} e \right) \right) & \text{for } \theta \in [1/2, 1); \\ \zeta (\theta, e) & \text{for } \theta = 1. \end{cases}$$

Then $$\zeta_t (0, e) = \overline{G} (F_t (e)) = G (F_t (e))$$ and $$\zeta_t (1, e) = e$$. Since $$h \in L^2 (\Omega) \cap L^\beta (\Omega)$$. By Lemma 4.5 (i) we have $$v_\lambda \in C (\Omega)$$. First, we claim that $$\lim_{\theta \to 1^-} \zeta_t (\theta, e) = e$$ and

$$\lim_{\theta \to 1^-} \zeta_t (\theta, e) = \overline{G} (\psi w (x + le)).$$

(a) $$\lim_{\theta \to 1^-} \zeta_t (\theta, e) = e$$ : since

$$\int_{\mathbb{R}^N} \frac{x}{|x|} \left( \left| \nabla \left[ \psi w \left( x + \frac{l}{2(1-\theta)} e \right) \right] \right|^2 + \left[ \psi w \left( x + \frac{l}{2(1-\theta)} e \right) \right]^2 \right) dx$$

$$= \int_{\mathbb{R}^N} \frac{x - \frac{l}{2(1-\theta)} e}{|x - \frac{l}{2(1-\theta)} e|} \left( |\nabla w (x)|^2 + |w (x)|^2 \right) dx + o(1)$$

$$= \left( \frac{2p}{p-2} \right) \alpha_0 (\mathbb{R}^N) e + o(1) \text{ as } \theta \to 1^-,$$

then $$\lim_{\theta \to 1^-} \zeta_t (\theta, e) = e$$.

(b) $$\lim_{\theta \to \frac{1}{2}^-} \zeta_t (\theta, e) = \overline{G} (\psi w (x + le))$$ : since $$\overline{G} \in C (\Theta, S^{N-1})$$, we obtain $$\lim_{\theta \to \frac{1}{2}^-} \zeta_t (\theta, l) = \overline{G} (\psi w (x + le))$$.

Thus, $$\zeta_t (\theta, e) \in C ([0, 1] \times S^{N-1}, S^{N-1})$$ and

$$\zeta_t (0, e) = G (F_t (e)) \text{ for all } e \in S^{N-1},$$

$$\zeta_t (1, e) = e \text{ for all } e \in S^{N-1},$$

provided $$\lambda \in (0, \lambda_*)$$ and $$l \in (l_*, \infty)$$. This completes the proof. \(\square\)

**Lemma 4.21** For $$\lambda \in (0, \lambda_*)$$ and $$l \in (l_*, \infty)$$, $$I_\lambda (u)$$ has at least two critical points in $$[I_\lambda < \alpha_\lambda + \alpha_0 (\mathbb{R}^N)]$$.

**Proof.** Applying Lemmas 4.15 and 4.20, we have for $$\lambda \in (0, \lambda_*)$$ and $$l \in (l_*, \infty)$$,

$$\text{cat} \left( [I_\lambda \leq \alpha_\lambda + \alpha_0 (\mathbb{R}^N) - \sigma_l] \right) \geq 2.$$

By Proposition 4.2, Lemma 4.14 and Lemma 4.16 (v), $$I_\lambda (u)$$ has at least two critical points in $$[I_\lambda < \alpha_\lambda + \alpha_0 (\mathbb{R}^N)]$$.

\(\square\)
We can now complete the proof of Theorem 1.3: For $\lambda \in (0, \lambda^*)$, from Theorem 4.10 and Lemma 4.21, the equation (1) has three positive solutions $v_\lambda, v_1, v_2$ such that $v_\lambda \in M^+_\lambda$ and $v_i \in M^-_\lambda$ for $i = 1, 2$. Then

\[ v_\lambda \neq v_i \text{ for } i = 1, 2. \]

Let $u_\lambda = \lambda^{\frac{1}{p-1}} v_\lambda$ and $u_i = \lambda^{\frac{1}{p-1}} v_i$, then $u_\lambda, u_1$ and $u_2$ are positive solutions of equation $(E_\lambda)$ and $u_\lambda \neq u_i$ for $i = 1, 2$. Thus, the equation $(E_\lambda)$ has at least three positive solutions.

Remark 4.3 If $\omega \subset B^N (0; \rho)$ and $\rho$ is sufficiently small such that Benci-Cerami’s minimax argument is holds for equation (2), then similar to the argument in Adachi-Tanaka [1], there exists a positive number $\lambda^* \leq \lambda_+$ such that Benci-Cerami’s minimax argument also work for equation (1) and $\lambda \in (0, \lambda^*)$. We can conclude that for $\lambda \in (0, \lambda^*)$, the equation $(E_\lambda)$ has at least four positive solutions since the critical value of our solutions in Theorem 1.3 are strictly lower than the first break down of the Palais–Smale condition. \hfill \blacksquare

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References


