The effect of domain shape on the number of positive and nodal solutions for semilinear elliptic equations

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Abstract
In this paper, we study the effect of domain shape on the number of positive and nodal (sign-changing) solutions for semilinear elliptic equations. We prove a semilinear elliptic equation in a domain $\Omega$ that contains $m$ disjoint large balls has $\frac{m^2}{2}$ nodal solutions in addition to $m$ positive solutions.

1 Introduction
In this paper, we study the multiplicity of positive and nodal solutions for the following semilinear elliptic equation:

$$
\begin{cases}
-\Delta u + u = |u|^{p-2}u^+ + |u|^{q-2}u^- \quad \text{in } \Omega, \\
u \in H^1_0(\Omega),
\end{cases} \quad (E_{p,q})
$$

where $\Omega$ is a domain in $\mathbb{R}^N$, $2 < p, q < 2^*(2^* = \frac{2N}{N-2} \text{ if } N \geq 3, \quad 2^* = \infty \text{ if } N = 2)$, $u^+ = \max\{0, u\}$, $u^- = u - u^+$ and $H^1_0(\Omega)$ is the Sobolev space in $\Omega$ with dual space $H^{-1}(\Omega)$. Associated with equation $(E_{p,q})$, we consider the energy functional $J$ in $H^1_0(\Omega)$

$$
J(u) = \frac{1}{2} \|u\|^2 - \frac{1}{p} \int_{\Omega} |u^+|^p - \frac{1}{q} \int_{\Omega} |u^-|^q
$$

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where \( \| u \| = \left( \int_{\Omega} |\nabla u|^2 + u^2 \right)^{1/2} \) is a standard norm in \( H^1_0(\Omega) \). It is well known that the functional \( J \in C^2( H^1_0(\Omega), \mathbb{R}) \) and the solutions of equation \( (E_{p,q}) \) are the critical points of the energy functional \( J \) in \( H^1_0(\Omega) \). (see Ambrosetti-Rabinowitz [1] and Willem [23]).

Generally, a standard technique to find the one sign solutions of equation \( (E_{p,q}) \) in \( \Omega \) is using the Nehari minimization problem:

\[
\alpha^\pm(\Omega) = \inf_{v \in \mathcal{M}^\pm(\Omega)} J(v),
\]

where \( \mathcal{M}^\pm(\Omega) = \{ u \in H^1_0(\Omega) \setminus \{0\} \mid \langle J'(u), u \rangle = 0, \pm u \geq 0 \} \). Note that \( \alpha^\pm(\Omega) \) are positive numbers (see Willem [23]). The existence of positive solutions of equation \( (E_{p,q}) \) is affected by the shape of the domain \( \Omega \) has been the focus of a great deal of research in recent years. By the Rellich compactness theorem, it is easy to obtain a positive solution of equation \( (E_{p,q}) \) in bounded domains. For general unbounded domains \( \Omega \), because of the lack of compactness, the existence of positive solutions of equation \( (E_{p,q}) \) in \( \Omega \) is very difficult and unclear. Indeed a classical, by now, result Esteban-Lions [17] states that, for a very large class of unbounded domains, those satisfying the condition: there exists \( \chi \in \mathbb{R}^N, ||\chi|| = 1 \) such that \( n(x) \cdot \chi \geq 0 \) and \( n(x) \cdot \chi \neq 0 \) on \( \partial \Omega \), where \( n(x) \) is the unit outward normal vector to \( \partial \Omega \) at the point \( x \). They asserted that equation \( (E_{p,q}) \) does not admit any nontrivial solution. Recently, there have been some progresses for the existence of positive solutions of equation \( (E_{p,q}) \) in unbounded domains as follows: Benci-Cerami [5] for \( \Omega \) is an exterior domain, Berestycki-Lions [6] for \( \Omega = \mathbb{R}^N \), Lien-Tzeng-Wang [18] for \( \Omega \) is a periodic domain, Chen-Wang [9] for \( \Omega \) is an interior flask domain, Del Pino-Felmer [13, 14] for \( \Omega \) is a quasicylindrical domain, Wu [24] for \( \Omega \) is a multi-bump domain and Wu [25] for \( \Omega \) is an unbounded dumbbell type domain.

In the aforementioned works, the authors considered positive solutions. For other situations, Bartsch [4] obtained infinite nodal solutions for equation \( (E_{p,q}) \) in bounded domains. Furtado [11, 12] showed that the domain topology is related with the number of 2–nodal solutions of equation \( (E_{p,q}) \). A 2–nodal solution is a nontrivial solution \( u \) such that the set \( \{ x \in \Omega \mid u(x) \neq 0 \} \) has exactly two connected components, \( u \) is positive in one of them and negative in the other (see Castro-Clapp [7] or Bartsch-Weth [3]). Bartsch-Weth [3], proved that equation \( (E_{p,q}) \) in a bounded domain \( \Omega \) that contains a large ball has three nodal solutions in which two 2–nodal solutions.

Motivated by the above results, we are interested in relation between the shape of domain and the number of positive and 2–nodal solutions of equation \( (E_{p,q}) \). Now, we state our main result in this paper. Take \( r > 0 \) and \( m \geq 1 \), we assume that the domains \( \Theta_0, \Theta_1(r), \Theta_2(r), \ldots, \Theta_m(r) \) are satisfying the following conditions:
(D1) there exists a positive number \( R_0 \) such that \( B^N (z; R_0) \setminus \Theta_0 \neq \emptyset \) for all \( z \in \mathbb{R}^N \), where \( B^N (z; r) = \{ x \in \mathbb{R}^N \mid |x - z| < r \} \);

(Dr) there exist points \( z_1, z_2, \ldots, z_m \) in \( \mathbb{R}^N \) such that

\[
B^N (z_i; r) \subseteq \Theta_i (r) \subseteq B^N (z_i; r + 1) \text{ for all } i \in \{1, 2, \ldots, m\},
\]

\(|z_i - z_j| > 3 (r + 1)\) and \( \Theta_0 \cup \bigcup_{i=1}^m \Theta_i (r) \) is a smooth domain in \( \mathbb{R}^N \).

Let \( \Theta (r) = \Theta_0 \cup \bigcup_{i=1}^m \Theta_i (r) \). Then we have the following result.

**Theorem 1.1** For each positive number \( \varepsilon \leq \min \left\{ \frac{p}{p-2} \alpha^+ (\mathbb{R}^N), \frac{q}{q-2} \alpha^- (\mathbb{R}^N) \right\} \) and unbounded domain \( \Theta_0 \) which satisfies the condition (D1), there exists \( r_0 > 0 \) such that if \( r \geq r_0 \) and the domains \( \Theta_0, \Theta_1 (r), \Theta_2 (r), \ldots, \Theta_m (r) \) are satisfying the condition (Dr), then equation \((E_{p,q})\) in \( \Theta (r) \) has \( m^2 \) \(2\)-nodal solutions \( \{u_{i,j}^0\}_{i,j \in \{1,2,\ldots,m\}} \) and \( m \) positive solutions \( u_0, u_0^2, \ldots, u_0^m \) with

\[
\int_{\Theta_i (r)} \left| (u_0^{i,j})^+ \right|^p < \varepsilon, \quad \int_{\Theta_j (r)} \left| (u_0^{i,j})^- \right|^q < \varepsilon \text{ for all } i, j \in \{1, 2, \ldots, m\},
\]

and

\[
\int_{\Theta_i (r)} |u_0|^p < \frac{\varepsilon}{2} \text{ for all } i \in \{1, 2, \ldots, m\},
\]

where \( (u_0^{i,j})^+ = \max \{ u_0^{i,j}, 0 \} \) and \( (u_0^{i,j})^- = u_0^{i,j} - (u_0^{i,j})^+ \).

By the condition (Dr) we have for \( i, j \in \{1, 2, \ldots, m\} \) with \( i \neq j \)

\[
dist (\Theta_i (r), \Theta_j (r)) \to \infty \text{ as } r \to \infty.
\]

Thus, if \( \Theta_0 \) is a bounded domain and \( m \geq 2 \), then the condition (Dr) cannot be use. Next, we modify conditions (D1) and (Dr) such that the result of Theorem 1.1 also holds even if \( \Theta_0 \) is a bounded domain.

(\( \overline{D1} \)) there exists a positive number \( b \) such that

\[
\Theta_0 \subset S (b) = \{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid -b < x_N < b \};
\]

(\( \overline{Dr} \)) there exist points \( z_1, z_2, \ldots, z_m \) in \( S (b) \) such that

\[
B^N (z_i; r) \subseteq \Theta_i (r) \subseteq B^N (z_i; r + 1) \text{ for all } i \in \{1, 2, \ldots, m\},
\]

\(|z_i - z_j| > 2 (r + 1)\) and \( \Theta_0 \cup \bigcup_{i=1}^m \Theta_i (r) \) is a smooth domain in \( \mathbb{R}^N \).
Furthermore, we have the following result which is the same in Theorem 1.1.

**Theorem 1.2** For each positive number \( \varepsilon \leq \min \left\{ \frac{p}{p-2}\alpha^{+}(\mathbb{R}^{N}), \frac{q}{q-2}\alpha^{-}(\mathbb{R}^{N}) \right\} \) and domain \( \Theta_0 \) which satisfies the condition \( (D_1) \), there exists \( r_0 > 0 \) such that if \( r \geq r_0 \) and the domains \( \Theta_0, \Theta_1(r), \Theta_2(r), \ldots, \Theta_m(r) \) are satisfying the condition \( (D_2) \), then equation \( (E_{p,q}) \) in \( \Theta(r) \) has \( m^2 \) \( 2 \)-nodal solutions \( \{u_{i,j}^{0}\}_{i,j \in \{1,2,\ldots,m\}} \) and \( m \) positive solutions \( u_1^0, u_2^0, \ldots, u_m^0 \) with

\[
\int_{\Theta_i^j(r)} \left| (u_{i,j}^0)^{+} \right|^p < \varepsilon; \quad \int_{\Theta_i^j(r)} \left| (u_{i,j}^0)^{-} \right|^q < \varepsilon \quad \text{for all } i, j \in \{1, 2, \ldots, m\},
\]

and

\[
\int_{\Theta_i^j(r)} \left| u_{i}^0 \right|^p < \frac{\varepsilon}{2} \quad \text{for all } i \in \{1, 2, \ldots, m\}.
\]

**Proof.** Similar to the argument in Theorem 1.1 and is omitted here. \( \Box \)

Among other interesting results, Del Pino-Felmer [15], Del Pino-Felmer-Wei [16], Noussair-Wei [20] and Wei [22] have considered the effect of domain topology on the existence of single–peak positive solutions, multi–peak positive solutions or nodal solutions. Roughly speaking, if \( \Omega \) has a "rich" topology, then the singular perturbation problem

\[
\begin{cases}
-\varepsilon \Delta u + u = |u|^{p-2}u & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]

has single–peak positive solutions, multi–peak positive solutions or nodal solutions provided that \( \varepsilon \) is sufficiently small.

This paper is organized as follows. In section 2, we describe various preliminaries. In section 3, we construct the Palais–Smale (simply by (PS)) sequences. In section 4, we prove Theorem 1.1.

## 2 Preliminaries

In this section, we recall several known results will be used for later sections. First, we define the (PS)–sequences in \( H_0^1(\Omega) \) for \( J \) as follows.

**Definition 2.1** For \( \beta \in \mathbb{R} \), a sequence \( \{u_n\} \) is a \( (PS)_\beta \)-sequence in \( H_0^1(\Omega) \) for \( J \) if \( J(u_n) = \beta + o(1) \) and \( J'(u_n) = o(1) \) strongly in \( H^{-1}(\Omega) \) as \( n \to \infty \).

For any \( \beta \in \mathbb{R} \), a \( (PS)_\beta \)-sequence in \( H_0^1(\Omega) \) for \( J \) is bounded.

**Lemma 2.2** Let \( \beta \in \mathbb{R} \) and \( \{u_n\} \) be a \( (PS)_\beta \)-sequence in \( H_0^1(\Omega) \) for \( J \), then there exists \( c > 0 \) such that \( \|u_n\| \leq c \) for all \( n \in \mathbb{N} \).
Let \( \xi \in C^{\infty}([0, \infty)) \) such that \( 0 \leq \xi \leq 1 \), \( \xi(t) = 0 \) for \( t \in [0, 1] \) and \( \xi(t) = 1 \) for \( t \in [2, \infty) \). For \( n \in \mathbb{N} \), we define
\[
\xi_n(x) = \xi\left(\frac{2|x|}{n}\right). \tag{1}
\]

Then we have the following useful lemma, whose proof can be found in Wu [24].

**Lemma 2.5** Suppose that \( \{u_n\} \) is a \((PS)_\beta\)-sequence in \( H^1_0(\Omega) \) for \( J \) satisfies \( u_n^\pm \rightharpoonup 0 \) weakly in \( H^1_0(\Omega) \). Let \( v_n^\pm = \xi_n u_n^\pm \). Then there exists a subsequence \( \{u_n\} \) such that \( \|u_n^\pm - v_n^\pm\| = o(1) \) as \( n \to \infty \). Furthermore,
(i) if \( u_n^+ \rightharpoonup 0 \) weakly in \( H^1_0(\Omega) \), then \( \|v_n^+\|^2 = \int_\Omega |v_n^+|^p + o(1) \);
(ii) if \( u_n^- \rightharpoonup 0 \) weakly in \( H^1_0(\Omega) \), then \( \|v_n^-\|^2 = \int_\Omega |v_n^-|^q + o(1) \).

**Proof.** See Proposition 4.1 in Bartsch-Weth [3]. □

**Lemma 2.6** If \( u \in H^1_0(\Omega) \) is a nodal solution of equation \((E_{p,q})\) in \( \Omega \) and \( J(u) < \theta(\Omega) + \min \{\alpha^+(\Omega), \alpha^-(\Omega)\} \), then \( u \) is a 2-nodal solution of equation \((E_{p,q})\) in \( \Omega \).

**Proof.** See Proposition 4.1 in Bartsch-Weth [3]. □
3 Palais–Smale Sequences

Throughout this section, we assume that the domains $\Theta_0, \Theta_1(r), \Theta_2(r), \ldots, \Theta_m(r)$ are satisfying conditions $(D1)$ and $(Dr)$. For each $i, j \in \{1, 2, \ldots, m\}$ and $0 < \varepsilon \leq \min \left\{ \frac{p}{p-2} \alpha^+ (\mathbb{R}^N), \frac{q}{q-2} \alpha^- (\mathbb{R}^N) \right\}$, we denote

$$M_i^+ (\varepsilon, r) = \left\{ u \in M^+ (\Theta (r)) \mid \int_{[\Theta_i(r)]^c} |u^+|^p < \varepsilon \right\} ;$$

$$\partial M_i^+ (\varepsilon, r) = \left\{ u \in M^+ (\Theta (r)) \mid \int_{[\Theta_i(r)]^c} |u^+|^p = \varepsilon \right\} ;$$

$$M_i^- (\varepsilon, r) = \left\{ u \in M^- (\Theta (r)) \mid \int_{[\Theta_i(r)]^c} |u^-|^q < \varepsilon \right\} ;$$

$$\partial M_i^- (\varepsilon, r) = \left\{ u \in M^- (\Theta (r)) \mid \int_{[\Theta_i(r)]^c} |u^-|^q = \varepsilon \right\} ;$$

$$N_{i,j} (\varepsilon, r) = \left\{ u \in H^1 (\Theta (r)) \mid u^+ \in M_i^+ (\varepsilon, r) \text{ and } u^- \in M_j^- (\varepsilon, r) \right\} ;$$

$$\partial N_{i,j} (\varepsilon, r) = \left\{ u \in H^1 (\Theta (r)) \mid u^+ \in \overline{M_i^+ (\varepsilon, r)}, u^- \in \overline{M_j^- (\varepsilon, r)} \text{ and either } u^+ \in \partial M_i^+ (\varepsilon, r) \text{ or } u^- \in \partial M_j^- (\varepsilon, r) \right\} ,$$

where $u^+ = \max \{ u, 0 \}$, $u^- = u - u^+$ and $M_i^\pm (\varepsilon, r)$ is a closure of $M_i^\pm (\varepsilon, r)$. It is easy to verify that $M_i^\pm (\varepsilon, r)$ and $N_{i,j} (\varepsilon, r)$ are nonempty sets for all $i, j \in \{1, 2, \ldots, m\}$. Note that, if $M_i^\pm (\varepsilon, r)$ and $N_{i,j} (\varepsilon, r)$ is denoted the closure of $M_i^\pm (\varepsilon, r)$ and $N_{i,j} (\varepsilon, r)$, respectively, then we have $M_i^\pm (\varepsilon, r) = M_i^\pm (\varepsilon, r) \cup \partial M_i^\pm (\varepsilon, r), \ N_{i,j} (\varepsilon, r) = N_{i,j} (\varepsilon, r) \cup \partial N_{i,j} (\varepsilon, r)$ and $\partial M_i^\pm (\varepsilon, r), \partial N_{i,j} (\varepsilon, r)$ is the boundary of $M_i^\pm (\varepsilon, r), N_{i,j} (\varepsilon, r)$, respectively. Furthermore, we have the following results.

**Lemma 3.1** For each $r \geq 2$, we have

(i) $N_{i,j} (\varepsilon, r)$ are disjoint;

(ii) $M_i^\pm (\varepsilon, r)$ and $M_j^\pm (\varepsilon, r)$ are disjoint for all $i \neq j$.

**Proof.** (i) Since the proof of all cases are similar. Thus, we only need to prove the case “1, 1” and “1, 2”. Assume the contrary, there exists $v_0 \in N_{1,1} (\varepsilon, r) \cap N_{1,2} (\varepsilon, r)$ such that

$$\int_{[\Theta_1(r)]^c} |v_0^-|^q < \varepsilon \text{ and } \int_{[\Theta_2(r)]^c} |v_0^-|^q < \varepsilon.$$
Since \( v_0 \in M^- (\Theta (r)) \), we have
\[
\frac{2q}{q - 2} \alpha^- (\Theta (r)) \leq \int_{\Theta (r)} |v_0|^q \leq \int_{|\Theta (r)|^c} |v_0|^q + \int_{|\Theta (r)|^c} |v_0^\uparrow|^q < \frac{2q}{q - 2} \alpha^- (\mathbb{R}^N),
\]
which is a contradiction.

The proof of (ii) is similar and omitted here. \( \square \)

Define the minimization problems in \( M^\pm_i (\frac{\varepsilon}{2}, r) \), \( \partial M^\pm_i (\frac{\varepsilon}{2}, r) \), \( N_i (\frac{\varepsilon}{2}, r) \) and \( \partial N_{i,j} (\frac{\varepsilon}{2}, r) \) for \( J \),
\[
\beta^\pm_i (r) = \inf_{v \in M^\pm_i (\frac{\varepsilon}{2}, r)} J (v); \quad \tilde{\beta}^\pm_i (r) = \inf_{v \in \partial M^\pm_i (\frac{\varepsilon}{2}, r)} J (v)
\]
and
\[
\gamma_{i,j} (r) = \inf_{v \in N_{i,j} (\frac{\varepsilon}{2}, r)} J (v); \quad \tilde{\gamma}_{i,j} (r) = \inf_{v \in \partial N_{i,j} (\frac{\varepsilon}{2}, r)} J (v).
\]

Clearly, \( \beta^\pm_i (r) \geq \alpha^\pm (\Theta (r)) \), \( \gamma_{i,j} (r) \geq \alpha^+ (\Theta (r)) + \alpha^- (\Theta (r)) \). Furthermore, we have the following results.

**Lemma 3.2** For each positive number \( \sigma < \min \{ \alpha^+ (\mathbb{R}^N), \alpha^- (\mathbb{R}^N) \} \) there exists \( r_0 > 0 \) such that
(i) \( \gamma_{i,j} (r) < \alpha^+ (\mathbb{R}^N) + \alpha^- (\mathbb{R}^N) + \sigma \);
(ii) \( \beta^\pm_i (r) < \alpha^\pm (\mathbb{R}^N) + \sigma \),
for all \( i, j \in \{ 1, 2, \ldots, m \} \) and \( r \geq r_0 \).

**Proof.** (i) Let \( x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \) and
\[
B^N_+ (0; r) = \{ x \in B^N (0; r) \mid x_N > 0 \} \quad \text{and} \quad B^N_- (0; r) = \{ x \in B^N (0; r) \mid x_N < 0 \}
\]
be half \( N \)-balls in \( \mathbb{R}^N \). By the Lien-Tzeng-Wang [18, Lemma 2.2],
\[
\alpha^\pm (B^N_+ (0; r)) \searrow \alpha^\pm (\mathbb{R}^N) \quad \text{as} \quad r \nearrow \infty.
\]
Thus, there exists \( r_1 > 0 \) such that \( \alpha^\pm (B^N_+ (0; r_1)) < \alpha^\pm (\mathbb{R}^N) + \frac{\sigma}{2} \). Moreover, by Ambrosetti-Rabinowitz [1], equation \( (E_{p,q}) \) in \( B^N_+ (0; r_1) \) and in \( B^N_- (0; r_1) \) has a positive solution \( v_+ \) and a negative solution \( v_- \), respectively, such that \( J (v_+) = \alpha^\pm (B^N_+ (0; r_1)) \). By Lien-Tzeng-Wang [18, Theorem 2.10], if \( \Omega \) is a domain in \( \mathbb{R}^N \), then \( \alpha^\pm (\Omega) \) is invariant by rigid motions. Thus,
\[
J (v_+) = \alpha^\pm ([B^N_+ (0; r_1) + x]) < \alpha^\pm (\mathbb{R}^N) + \frac{\sigma}{2} \quad \text{for all} \ x \in \mathbb{R}^N.
\]
Set \( v_i(x) = u_+ (x - z_i) \) and \( v_j(x) = u_- (x - z_j) \). Clearly, \( v_i \in M^+ (\Theta (r)) \), \( v_j \in M^- (\Theta (r)) \) and
\[
\int_{[B^N(z_i; r1)]^c} |v_i^+|^p = \int_{[B^N(z_j; r1)]^c} |v_j^-|^q = 0.
\]
Thus, \( v_i \in M^+_i \left( \frac{\varepsilon}{2}, r \right) \) and \( v_j \in M^-_j \left( \frac{\varepsilon}{2}, r \right) \) for all \( r \geq r_1 \). Set \( v_{i,j} = v_i + v_j \), we obtain \( v_{i,j} \in N_{i,j} \left( \frac{\varepsilon}{2}, r \right) \) and
\[
\gamma_{i,j} (r) \leq J (v_{i,j}) < \alpha^+ (\mathbb{R}^N) + \alpha^- (\mathbb{R}^N) + \sigma
\]
for all \( i, j \in \{1, 2, \ldots, m\} \) and \( r \geq r_1 \).

The proof of (ii) is similar and omitted here.

**Lemma 3.3** There exist positive numbers \( \delta, r_2 \) such that for each \( i, j \in \{1, 2, \ldots, m\} \),
(i) \( \tilde{\gamma}_{i,j} (r) > \alpha^+ (\mathbb{R}^N) + \alpha^- (\mathbb{R}^N) + \delta \) for all \( r \geq r_2 \);
(ii) \( \tilde{\beta}_i (r) > \alpha^\pm (\mathbb{R}^N) + \delta \) for all \( r \geq r_2 \).

**Proof.** (i) Fix \( i, j \in \{1, 2, \ldots, m\} \). Assume the contrary, there exist \( \{r_n\} \subset \mathbb{R}^+ \) with \( r_n \to \infty \) as \( n \to \infty \), \( \{z_{i,n}\}, \{z_{j,n}\} \subset \mathbb{R}^N \) and \( \{u_n\} \subset \partial N_{i,j} \left( \frac{\varepsilon}{2}, r_n \right) \) such that
\[
J (u_n) \to c \leq 2\alpha^+ (\mathbb{R}^N) + \alpha^- (\mathbb{R}^N), \quad (2)
\]
\[
\int_{\Theta(r_n)} |\nabla u_n^+|^2 + (u_n^+)^2 = \int_{\Theta(r_n)} |u_n^+|^p, \quad (3)
\]
\[
\int_{\Theta(r_n)} |\nabla u_n^-|^2 + (u_n^-)^2 = \int_{\Theta(r_n)} |u_n^-|^q, \quad (4)
\]
and either \( u_n^+ \in \partial M^+_i \left( \frac{\varepsilon}{2}, r_n \right) \) or \( u_n^- \in \partial M^-_j \left( \frac{\varepsilon}{2}, r_n \right) \). Since \( J (u_n) = J (u_n^+) + J (u_n^-) \) and \( J (u_n^+) \geq \alpha^\pm (\mathbb{R}^N) \). Thus, \( J (u_n^+) \to \alpha^\pm (\mathbb{R}^N) \). Moreover, by (3), (4) and Lemma 2.4, \( \{u_n^\pm\} \) is \((PS)_{\alpha^\pm (\mathbb{R}^N)}\)-sequences in \( H^1 (\mathbb{R}^N) \) for \( J \). By (3) and the Sobolev imbedding theorem, there exists \( d > 0 \) such that \( \int_{\Theta(r_n)} |\nabla u_n^+|^2 + (u_n^+)^2 > d \) for all \( n \). From the concentration compactness principle of Lions [19], there exist positive numbers \( R \), \( d \) and \( \{y_n^\pm\} \subset \mathbb{R}^N \) such that
\[
\int_{B^N(y_n^+; R)} |u_n^+|^q \geq d \text{ and } \int_{B^N(y_n^-; R)} |u_n^-|^q \geq d \text{ for all } n.
\]
Without loss of generality, we may assume that \( u_n^+ \in \partial M^+_i \left( \frac{\varepsilon}{2}, r_n \right) \), that is
\[
\int_{\Theta_i(r_n)} |u_n^+|^p = \frac{\varepsilon}{2} \leq \frac{1}{2} \min \left\{ \frac{p}{p - 2} \alpha^+ (\mathbb{R}^N), \frac{q}{q - 2} \alpha^- (\mathbb{R}^N) \right\}.
\]
Let $\tilde{u}_n(x) = u^+_n(x + y^+_n)$. Then $\tilde{u}_n \in M^+ (R^N)$ and there is a $u_0 \in H^1 (R^N)$ such that

$$\tilde{u}_n \rightharpoonup u_0 \text{ weakly in } H^1 (R^N) \text{ as } n \to \infty,$$

$$\tilde{u}_n \rightarrow u_0 \text{ a.e. in } R^N \text{ as } n \to \infty$$

and

$$\int_{B^N(0;R)} |\tilde{u}_n|^p \rightarrow \int_{B^N(0;R)} |u_0|^p \geq d \text{ as } n \to \infty.$$

Moreover, by Bahri-Lions [2] and the strong maximum principle

$$\tilde{u}_n \rightarrow u_0 \text{ strongly in } H^1 (R^N) \text{ as } n \to \infty,$$

$u_0$ is a positive solution of equation $(E_{p,q})$ in $R^N$ and $J(u_0) = \alpha^+ (R^N)$. Now, we consider the sequence $\{z_i,n - y^+_n\}$. By passing to a subsequence if necessary, we may assume that one of the following cases occurs:

case (I) $\{z_i,n - y^+_n\}$ is bounded;

case (II) $\{z_i,n - y^+_n\}$ is unbounded and for each $R > 0$ there exists $n(R) \in N$ such that

$$B^N (0; R) \cap [\Theta_i(r_n) - y^+_n] = \emptyset \text{ for all } n \geq n(R);$$

case (III) $\{z_i,n - y^+_n\}$ is unbounded and there exists $R_1 > 0$ such that

$$B^N (0; R_1) \cap [\Theta_i(r_n) - y^+_n] \neq \emptyset \text{ for all } n.$$

Since $\|\tilde{u}_n - u_0\| \to 0$ as $n \to \infty$. By the Sobolev imbedding theorem and the Vitali convergence theorem, there exists $R (\frac{\varepsilon}{2}) > R_0$ such that

$$\int_{|z| > R (\frac{\varepsilon}{2})} |\tilde{u}_n|^p < \frac{\varepsilon}{2} \text{ for all } n.$$

In case (I) : We may assume $z_i,n - y^+_n \rightarrow z_0$. By $r_n \to \infty$ as $n \to \infty$, there exists $n_0 \in N$ such that $B^N (0; R (\frac{\varepsilon}{2})) \subset [\Theta_i(r_n) - y^+_n]$ for all $n \geq n_0$. Thus, for each $n \geq n_0$

$$\int_{[\Theta_i(r_n)]} |u^+_n|^p = \int_{[\Theta_i(r_n) - y^+_n]} |\tilde{u}_n|^p \leq \int_{|z| > R (\frac{\varepsilon}{2})} |\tilde{u}_n|^p < \frac{\varepsilon}{2},$$

this contradicts (5).

In case (II) : By the hypothesis, there exists $n_0 = n (R (\frac{\varepsilon}{2})) \in N$ such that

$$B^N (0; R (\frac{\varepsilon}{2})) \cap [\Theta_i(r_n) - y^+_n] = \emptyset \text{ for all } n \geq n_0.$$

Thus,

$$\int_{[\Theta_i(r_n) - y^+_n]} |\tilde{u}_n|^p \leq \int_{|z| > R (\frac{\varepsilon}{2})} |\tilde{u}_n|^p < \frac{\varepsilon}{2} \text{ for all } n \geq n_0. \quad (7)$$
Since \( \{u_n^+\} \subset M^+ (\Theta (r_n)) \), this means
\[
\int_{\Theta (r_n)} |u_n^+|^p > \frac{2p}{p-2} \alpha^+ (\mathbb{R}^N) \quad \text{for all } n. \tag{8}
\]
From (7) and (8), we obtain
\[
\int_{[\Theta_i (r_n)]^c} |u_n^+|^p = \int_{\Theta (r_n)} |u_n^+|^p - \int_{[\Theta_i (r_n) - y_n^+]} |\bar{u}_n|^p > \frac{3p}{2 (p-2)} \alpha^+ (\mathbb{R}^N)
\]
for all \( n \geq n_0 \), this contradicts (5).

In case (III) : First, we claim that for each \( R \geq \max \{ R_1, R(\frac{\varepsilon}{2}) \} \) there exists \( n (R) \in \mathbb{N} \) such that
\[
B^N (0; R) \cap [\Theta_i (r_n) - y_n^+] = \emptyset \tag{9}
\]
for all \( j \in \{1, 2, \ldots, i-1, i+1, \ldots, m \} \) and \( n \geq n (R) \). By the condition (D1),
\[
B^N (0; R) \setminus [\bar{\Theta}_0 - y_n^+] \neq \emptyset \quad \text{for all } n. \tag{10}
\]
Since \( \bar{u}_n \equiv 0 \) in \( [\Theta (r_n) - y_n^+]^c \),
\[
\bar{u}_n \to u_0 \text{ a.e. in } \mathbb{R}^N \text{ as } n \to \infty
\]
and \( u_0 \) is a positive solution of equation \((E_{p,q})\) in \( \mathbb{R}^N \), we have
\[
\lim_{n \to \infty} [\Theta (r_n) - y_n^+] = \mathbb{R}^N. \tag{11}
\]
Since \( |z_{i,n} - z_{j,n}| > 3 (r_n + 1) \) and \( r_n \to \infty \) as \( n \to \infty \), there exists \( n (R) \in \mathbb{N} \) such that
\[
\text{dist} (\Theta_i (r_n), \Theta_j (r_n)) > 2R \quad \text{for all } i \neq j \text{ and } n \geq n (R). \tag{12}
\]
Moreover,
\[
B^N (0; R) \cap [\Theta_i (r_n) - y_n^+] = \emptyset \quad \text{for all } n.
\]
Therefore, (9) holds. By (9) and (11), for each \( R \geq \max \{ R_1, R(\frac{\varepsilon}{2}) \} \) there exists \( \pi \in \mathbb{N} \) such that
\[
B^N (0; R) \subseteq [\Theta_i (r_n) - y_n^+] \quad \text{for all } n \geq \pi. \tag{12}
\]
From (6) and (12), we can conclude that for \( n \geq \pi \)
\[
\int_{\Theta_i (r_n)} |u_n^+|^p = \int_{[\Theta_i (r_n) - y_n^+]} |\bar{u}_n|^p \geq \int_{|z| \leq R (\frac{\varepsilon}{2})} |\bar{u}_n|^p > \frac{2p}{p-2} \alpha^+ (\mathbb{R}^N) - \frac{\varepsilon}{2}
\]
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or
\[
\int_{[\Theta_i(r_n)]^c} |u_n^+|^p < \frac{\varepsilon}{2},
\]
this contradicts (5).

The proof of (ii) is similar and omitted here.

By Lemmas 3.2, 3.3, there exists \( r_0 > 0 \) such that for \( r > r_0 \)
\[
\gamma_{i,j}(r) < \min\{\alpha^+(\mathbb{R}^N) + \alpha^-(\mathbb{R}^N) + \min\{\alpha^+(\mathbb{R}^N), \alpha^-(\mathbb{R}^N)\}, \tilde{\gamma}_{i,j}(r)\}
\]
and
\[
\beta_{i}^{\pm}(r) < \min\{\alpha^+(\mathbb{R}^N) + \alpha^-(\mathbb{R}^N), \tilde{\beta}_{i}^{\pm}(r)\}
\]
for all \( i, j \in \{1, 2, \ldots, m\} \). Furthermore, we will use the idea of Bartsch-Weth [3] and Clapp-Weth [10] to get the following results.

**Lemma 3.4** There exists \( \mu_0 > 0 \) such that, for each \( v \in N_{i,j}\left(\frac{\varepsilon}{2}, r\right) \) and \( u \in H^1_0(\Theta(r)) \) with \( \|v - u\| < \mu_0 \), we have
\[
\left|\int_{[\Theta_i(r)]^c} |u^+|^p - \int_{[\Theta_i(r)]^c} |v^+|^p\right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left|\int_{[\Theta_j(r)]^c} |u^-|^q - \int_{[\Theta_j(r)]^c} |v^-|^q\right| < \frac{\varepsilon}{2}.
\]

**Proof.** If not, then there exist \( \{v_n\} \subset N_{i,j}\left(\frac{\varepsilon}{2}, r\right) \) and \( \{u_n\} \subset H^1_0(\Theta(r)) \) such that \( \|v_n - u_n\| \to 0 \), but
\[
\left|\int_{[\Theta_i(r)]^c} |u_n^+|^p - \int_{[\Theta_i(r)]^c} |v_n^+|^p\right| \geq \frac{\varepsilon}{2} \quad \ldots (13)
\]
or
\[
\left|\int_{[\Theta_j(r)]^c} |u_n^-|^q - \int_{[\Theta_j(r)]^c} |v_n^-|^q\right| \geq \frac{\varepsilon}{2}. \quad \ldots (14)
\]

Since
\[
\int_{[\Theta_i(r)]^c} |u_n^+ - v_n^+|^p \leq \int_{\Theta(r)} |u_n^+ - v_n^+|^p \leq \int_{\Theta(r)} |u_n - v_n|^p \to 0
\]
and
\[
\int_{[\Theta_j(r)]^c} |u_n^- - v_n^-|^q \leq \int_{\Theta(r)} |u_n^- - v_n^-|^q \leq \int_{\Theta(r)} |u_n - v_n|^q \to 0.
\]

Thus, by the Minkowski inequality
\[
\left|\int_{[\Theta_i(r)]^c} |u_n^+|^p - \int_{[\Theta_i(r)]^c} |v_n^+|^p\right| \to 0 \quad \text{and} \quad \left|\int_{[\Theta_j(r)]^c} |u_n^-|^q - \int_{[\Theta_j(r)]^c} |v_n^-|^q\right| \to 0
\]
this contradicts (13) and (14). \( \square \)
Lemma 3.5 For each $v_0 \in N_{i,j} \left( \frac{\varepsilon}{2}, r \right)$ there exists a map $h : H^1_0(\Theta(\cdot)) \to \mathbb{R}^2$ such that

(i) $h\left(s_1v_0^+ + s_2v_0^-\right) = (s_1, s_2)$ for $s_1, s_2 \geq 0$;

(ii) $h(u) = (1, 1)$ if and only if $u \in N_{i,j} \left( \frac{\varepsilon}{2}, r \right)$.

Proof. Similar to the method used in Clapp-Weth [10, Lemma 13]. \qed

Let $b = \alpha^+ (\mathbb{R}^N) + \alpha^- (\mathbb{R}^N) + \min \left\{ \alpha^+ (\mathbb{R}^N), \alpha^- (\mathbb{R}^N) \right\}$. Then we have the following results.

Proposition 3.6 Let $\lambda_0 = b - \gamma_{i,j} (r)$. Then for each $\lambda \in (0, \lambda_0)$ and $\mu \in (0, \mu_0)$ there exists $u_0 \in H^1_0(\Theta(\cdot))$ such that

(i) $\text{dist} \left( u_0, N_{i,j} \left( \frac{\varepsilon}{2}, r \right) \right) \leq \mu$;

(ii) $J(u_0) \in \left[ \gamma_{i,j}(r), \gamma_{i,j}(r) + \lambda \right]$;

(iii) $\| \nabla J(u_0) \| \leq \max \left\{ \sqrt{\lambda}, \frac{1}{\mu} \right\}$;

(iv) $\int_{\Theta(r)^c} |u_0^+|^p < \varepsilon$ and $\int_{\Theta(r)^c} |u_0^-|^q < \varepsilon$.

Proof. Fix $v_0 \in N_{i,j} \left( \frac{\varepsilon}{2}, r \right)$ such that $J(v_0) < \gamma_{i,j}(r) + \lambda$, and fix $l_0 > 1$ such that $J\left(l_0v_0^\pm\right) \leq 0$. Let $h : H^1_0(\Theta(\cdot)) \to \mathbb{R}^2$ as in Lemma 3.5. We put $K = [0, l_0] \times [0, l_0]$ and define

$$\eta : K \to H^1_0(\Theta(\cdot)), \quad \eta(s_1, s_2) = s_1v_0^+ + s_2v_0^-.$$ 

Then $h \circ \eta = \text{id} : K \to K$, in particular

$$\deg (h \circ \eta, K, (1, 1)) = 1. \quad (15)$$

Notice also that

$$J(\eta(s_1, s_2)) \leq J(v_0) < \gamma_{i,j}(r) + \lambda \quad \text{for all } (s_1, s_2) \in K. \quad (16)$$

Now we choose a Lipschitz continuous function $\chi : \mathbb{R} \to \mathbb{R}$ such that $0 \leq \chi \leq 1, \chi(s) = 1$ for $s \geq 0$ and $\chi(s) = 0$ for $s \leq -1$. Since $J \in C^2(H^1_0(\Theta(\cdot)), \mathbb{R})$, there is a semiflow $\varphi : [0, \infty) \times H^1_0(\Theta(\cdot)) \to H^1_0(\Theta(\cdot))$ satisfying

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \varphi(t, u) = -\chi(J(\varphi(t, u))) \nabla J(\varphi(t, u)), \\
\varphi(0, u) = u.
\end{array} \right.$$ 

We will frequently write $\varphi^t$ in place of $\varphi(t, \cdot)$. Since

$$J(v_0^\pm) < \gamma_{i,j}(r) + \lambda - \alpha^\pm (\mathbb{R}^N) < \alpha^+ (\mathbb{R}^N) + \alpha^- (\mathbb{R}^N) \quad \text{and} \quad J(l_0v_0^\pm) \leq 0,$$

it follows that

$$\sup J(\eta(\partial K)) < \alpha^+ (\mathbb{R}^N) + \alpha^- (\mathbb{R}^N).$$

Hence

$$(\varphi^t \circ \eta)(\partial K) \cap N(\Theta(\cdot)) = \emptyset \quad \text{for all } t \geq 0.$$
By Lemma 3.5, this implies
\[(h \circ \varphi^t \circ \eta) (y) \neq (1, 1) \text{ for all } y \in \partial K, t \geq 0.\]

Equality (15) and the global continuation principle of Leray-Schauder (see e.g. Zeider [27, p.629]) imply that there exists a connected subset \(Z \subset K \times [0, 1]\) such that
\[(1, 1, 0) \in Z; \quad \varphi^t (\eta (s_1, s_2)) \in N (\Theta (r)) \text{ for all } (s_1, s_2, t) \in Z; \quad Z \cap (K \times \{1\}) \neq \emptyset.\]

We put
\[\tilde{Z} = \{ \varphi^t (\eta (s_1, s_2)) \in N (\Theta (r)) \mid (s_1, s_2, t) \in Z \} .\]

By inequality (16),
\[\sup_{u \in \tilde{Z}} J (u) < \gamma_{i,j} (r) + \lambda < b.\]

Therefore, since \(Z\) is connected, we obtain that \(\tilde{Z} \subset N_{i,j} (\frac{r}{2}, r)\). Now we pick \((\bar{s}_1, \bar{s}_2, 1) \in Z \cap (K \times \{1\})\) and write
\[v_1 := \eta (\bar{s}_1, \bar{s}_2), v_2 := \varphi^t (v_1) .\]

Then \(v_2 \in \tilde{Z} \subset N_{i,j} (\frac{r}{2}, r)\). We distinguish two cases.

Case 1. \(||\varphi^t (v_1) - v_2|| \leq \mu\) for all \(t \in [0, 1]\). Then by Lemma 3.4, we have
\[\int_{[\Theta_i (r)]^c} \left| \varphi^t (v_1) \right|^p < \varepsilon \quad \text{and} \quad \int_{[\Theta_j (r)]^c} \left| \varphi^t (v_1) \right|^{q} < \varepsilon \]
for all \(t \in [0, 1]\). We choose \(t_0 \in [0, 1]\) with
\[\|\nabla J (\varphi^{t_0} (v_1))\| = \min_{0 \leq t \leq 1} \|\nabla J (\varphi^t (v_1))\|\]
and put \(u_0 = \varphi^{t_0} (v_1)\). Thus,
\[\lambda \geq J (v_1) - J (v_2) = - \int_0^1 \frac{\partial}{\partial t} J (\varphi^t (v_1)) dt \]
\[= \int_0^1 \|\nabla J (\varphi^t (v_1))\|^2 dt \geq \|\nabla J (u_0)\|^2.\]

We obtain \(u_0\) has the desired properties.

Case 2. There exists \(\bar{t} \in [0, 1]\) such that \(||\varphi^{\bar{t}} (v_1) - v_2|| > \mu\). Then let
\[t_1 = \sup \{ t \geq \bar{t} \mid \|\varphi^t (v_1) - v_2\| > \mu \} .\]
By Lemma 3.4,
\[
\int_{[\Theta_i(r)]^c} |(\varphi^t(v_1))^+|^p < \varepsilon \quad \text{and} \quad \int_{[\Theta_j(r)]^c} |(\varphi^t(v_1))^-|^q < \varepsilon
\]
for all \( t \in [t_1, 1] \). We choose \( t_0 \in [t_1, 1] \) with
\[
\|\nabla J(\varphi^{t_0}(v_1))\| = \min_{t_1 \leq t \leq 1} \|\nabla J(\varphi^t(v_1))\|
\]
and put \( u_0 = \varphi^{t_0}(v_1) \). Then
\[
\mu \leq \int_{t_1}^1 \left\| \frac{\partial}{\partial t} \varphi^t(v_1) \right\| dt \leq \int_{t_1}^1 \|\nabla J(\varphi^t(v_1))\| dt
\]
and
\[
\lambda \geq J(\varphi^{t_1}(v_1)) - J(v_2) = \int_{t_1}^1 \|\nabla J(\varphi^t(v_1))\|^2 dt \geq \|\nabla J(u_0)\| \int_{t_1}^1 \|\nabla J(\varphi^t(v_1))\| dt.
\]
We conclude that \( \|\nabla J(u_0)\| \leq \frac{\lambda}{\mu} \). Thus, \( u_0 \) has the desired properties. \( \square \)

**Corollary 3.7** For each \( r > r_0 \) there exists a sequence \( \{u_n^{i,j}\} \subset H^1_0(\Theta(r)) \) such that
1. \( \text{dist}(u_n^{i,j}, N_{i,j}(\frac{\varepsilon}{2}, r)) \to 0 \);
2. \( J(u_n^{i,j}) \to \gamma_{i,j}(r) < \alpha^+ + \alpha^-(R^N) + \min \{\alpha^+(R^N), \alpha^-(R^N)\} \); 
3. \( J'(u_n^{i,j}) = o(1) \) strongly in \( H^{-1}(\Theta(r)) \); 
4. \( \int_{[\Theta_i(r)]^c} \left| (u_n^{i,j})^+ \right|^p < \varepsilon \) and \( \int_{[\Theta_j(r)]^c} \left| (u_n^{i,j})^- \right|^q < \varepsilon \).

For the set \( M_i(\frac{\varepsilon}{2}, r) \), by a similar argument in Wu [24] (or see Cao-Noussair [8]), we have

**Proposition 3.8** For each \( r > r_0 \) there exists a sequence \( \{u_n^i\} \subset M_i(\frac{\varepsilon}{2}, r) \) such that \( J(u_n^i) \to \beta^+ \) \( \to \beta^-(r) < \min \{2\alpha^+(R^N), \beta_i(r)\} \) and \( J'(u_n^i) = o(1) \) strongly in \( H^{-1}(\Theta(r)) \).

### 4 Proof of Theorem 1.1

In this section we establish the existence of \( m \) positive solutions and \( m^2 \) 2–nodal solutions of equation \( (E_{p,q}) \) in \( \Theta(r) \) provided that \( r \) is sufficiently large. For each \( i, j \in \{1, 2, \ldots, m\} \) and \( r > r_0 \), we have the following compactness results.
Proposition 4.1 For each sequence \( \{u_{i,j}^n\} \subset H^1_0(\Theta (r)) \) which satisfies

\( (i) \) \( \text{dist} (u_{i,j}^n, N_{i,j} (\xi, r)) \to 0; \)
\( (ii) \) \( J (u_{i,j}^n) \to \gamma_{i,j} (r); \)
\( (iii) \) \( J' (u_{i,j}^n) = o(1) \) strongly in \( H^{-1}(\Theta (r)); \)
\( (iv) \) \( \int_{\Theta (r)}^c \left| (u_{i,j}^n)^+ \right|^p < \varepsilon \) and \( \int_{\Theta (r)}^c \left| (u_{i,j}^n)^- \right|^q < \varepsilon, \)

there exist a subsequence \( \{u_{i,j}^n\} \) and \( u_{i,j}^0 \in N_{i,j} (\varepsilon, r) \) such that \( u_{i,j}^n \to u_{i,j}^0 \) strongly in \( H^1_0(\Theta (r)). \) Furthermore, \( u_{i,j}^0 \) is a 2-nodal solution of equation \( (E_{p,q}) \) in \( \Theta (r). \)

**Proof.** Since \( \{u_{i,j}^n\} \) is bounded in \( H^1_0(\Theta (r)), \) we have \( \{u_{i,j}^n\}^+ \) and \( \{u_{i,j}^n\}^- \) are also bounded in \( H^1_0(\Theta (r)) \) and

\[
\left\| (u_{i,j}^n)^+ \right\|^2 = \int_{\Theta (r)} \left| (u_{i,j}^n)^+ \right|^p + o(1) \quad \text{and} \quad \left\| (u_{i,j}^n)^- \right\|^2 = \int_{\Theta (r)} \left| (u_{i,j}^n)^- \right|^q + o(1).
\]

Thus, there exist a subsequence \( \{u_{i,j}^n\} \) and \( u_{i,j}^0 \) in \( H^1_0(\Theta (r)) \) such that

\[
\left.\begin{array}{l}
u_{i,j}^n \rightharpoonup u_{i,j}^0; (u_{i,j}^n)^\pm \rightharpoonup (u_{i,j}^0)^\pm \text{ weakly in } H^1_0(\Theta (r)) \\
u_{i,j}^n \to u_{i,j}^0; (u_{i,j}^n)^\pm \to (u_{i,j}^0)^\pm \text{ a.e. in } \Theta (r).
\end{array}\right\}
\]

(17)

Moreover, \( u_{i,j}^0 \) is a solution of equation \( (E_{p,q}) \) in \( \Theta (r). \) We will show that \( (u_{i,j}^0)^\pm \neq 0. \) If not, we may assume that \( (u_{i,j}^0)^+ \equiv 0 \) and \( J (u_{i,j}^0)^+ = c + o(1) \) for some \( c > 0. \) By Lemma 2.5, there exists a subsequence \( \{u_{i,j}^n\} \) such that \( J (\xi_n u_{i,j}^n) = c + o(1) \) and

\[
\left\| \xi_n (u_{i,j}^n)^+ \right\|^2 = \int_{\Theta (r)} \left| \xi_n (u_{i,j}^n)^+ \right|^p + o(1),
\]

where \( \xi_n \) is as in \( (1). \) Moreover, by Lemma 2.3, there exist sequences \( \{s_{i,j}^n\} \subset \mathbb{R}^+ \setminus \{0\} \) with \( s_{i,j}^n = 1 + o(1) \) such that

\[
J \left( s_{i,j}^n \xi_n (u_{i,j}^n)^+ \right) = c + o(1)
\]

and

\[
\left\| s_{i,j}^n \xi_n (u_{i,j}^n)^+ \right\|^2 = \int_{\Theta (r)} \left| s_{i,j}^n \xi_n (u_{i,j}^n)^+ \right|^p.
\]

Then there exists \( n_0 \in \mathbb{N} \) such that for \( n > 2n_0, \)

\[
\frac{2p}{p-2} \alpha^+ (\Theta (r)) \leq \int_{\Theta (r)} \left| s_{i,j}^n \xi_n (u_{i,j}^n)^+ \right|^p = \int_{\Theta (r)} \left| (u_{i,j}^n)^+ \right|^p + o(1) < \frac{p}{p-2} \alpha^+ (\mathbb{R}^N) + o(1)
\]

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which is a contradiction. Therefore, \((u_{0}^{i,j})^{\pm} \neq 0\) and \(u_{0}^{i,j}\) is a nodal solution of equation \((E_{p,q})\) in \(\Theta (r)\). By the Fatou lemma, we have
\[
\int_{[\Theta_{i}(r)]^{c}} \left| (u_{0}^{i,j})^{+} \right|^{p} \leq \lim \inf \int_{[\Theta_{i}(r)]^{c}} \left| (u_{n}^{i,j})^{+} \right|^{p} < \varepsilon,
\]
\[
\int_{[\Theta_{j}(r)]^{c}} \left| (u_{0}^{i,j})^{-} \right|^{q} \leq \lim \inf \int_{[\Theta_{j}(r)]^{c}} \left| (u_{n}^{i,j})^{-} \right|^{q} < \varepsilon,
\]
and
\[
J (u_{0}^{i,j}) = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Theta(r)} \left| (u_{0}^{i,j})^{+} \right|^{p} + \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Theta(r)} \left| (u_{0}^{i,j})^{-} \right|^{q}
\]
\[
\leq \lim \inf \left[ \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Theta(r)} \left| (u_{n}^{i,j})^{+} \right|^{p} + \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Theta(r)} \left| (u_{n}^{i,j})^{-} \right|^{q} \right]
\]
\[
= \gamma_{i,j} (r).
\]
Thus, \(u_{0}^{i,j} \in N_{i,j} (\varepsilon, r)\) and \(J (u_{0}^{i,j}) = \gamma_{i,j} (r)\). Moreover, by the concentration compactness principle of Lions [19] and
\[
\gamma_{i,j} (r) < \alpha^{+} (\mathbb{R}^{N}) + \alpha^{-} (\mathbb{R}^{N}) + \min \left\{ \alpha^{+} (\mathbb{R}^{N}), \alpha^{-} (\mathbb{R}^{N}) \right\},
\]
we have \(u_{n}^{i,j} \rightarrow u_{0}^{i,j}\) strongly in \(H_{0}^{1}(\Theta (r))\) and \(u_{0}^{i,j}\) is a 2-nodal solution of equation \((E_{p,q})\) in \(\Theta (r)\).

For the set \(M_{i} \left( \frac{\varepsilon}{2}, r \right)\), by a similar argument, we have

**Proposition 4.2** For each sequence \(\{u_{n}^{i,j}\} \subset M_{i}^{+} \left( \frac{\varepsilon}{2}, r \right)\) which satisfies \(J (u_{n}^{i}) \rightarrow \beta_{i}^{+} (r)\) and \(J'(u_{n}^{i}) = o(1)\) strongly in \(H^{-1} (\Theta (r))\) there exist a subsequence \(\{u_{n}^{i}\}\) and \(u_{0}^{i} \in M_{i}^{+} \left( \frac{\varepsilon}{2}, r \right)\) such that \(u_{n}^{i} \rightarrow u_{0}^{i}\) strongly in \(H_{0}^{1}(\Theta (r))\). Furthermore, \(u_{0}^{i}\) is a positive solution of equation \((E_{p,q})\) in \(\Theta (r)\) and \(J (u_{0}^{i}) = \beta_{i}^{+} (r)\).

Now, we begin to show the proof of Theorem 1.1: By Corollary 3.7 and Propositions 3.8, 4.1, 4.2, equation \((E_{p,q})\) in \(\Theta (r)\), there exist 2-nodal solutions \(u_{n}^{i,j} \in N_{i,j} (\varepsilon, r)\) and positive solutions \(u_{0}^{i} \in M_{i}^{+} \left( \frac{\varepsilon}{2}, r \right)\). By Lemma 3.1, \(u_{0}^{i,j} \neq u_{0}^{i,k}\) for all \(i, j, k \in \{1, 2, \ldots, m\}\) with \(j \neq k\) and \(u_{0}^{i} \neq u_{0}^{j}\) for all \(i, j \in \{1, 2, \ldots, m\}\) with \(i \neq j\). Thus, we have proved Theorem 1.1.

**References**


