Existence and multiplicity of positive solutions for a class of nonlinear boundary value problems

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Abstract
In this paper, we study a class of nonlinear boundary value problems in $\mathbb{R}^N_+$. By means of minimax method and the Lusternik-Schnirelman category, the criteria of the existence, multiplicity and nonexistence of positive solutions are established.

Keywords: Lusternik-Schnirelmann category; Positive solutions; Nonlinear boundary value problems.

1. Introduction
Let $\Omega$ be a domain in $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial \Omega$ and consider the following nonlinear boundary value problem:

$$
-\Delta u = g(x,u) \text{ in } \Omega, \\
\frac{\partial u}{\partial n} + f(x,u) = 0 \text{ in } \partial \Omega,
$$

where $\frac{\partial}{\partial n}$ is the outer unit normal derivative, $g: \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $f: \partial \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Equations of the type (1.1) arise in many and diverse contexts like differential geometry (e.g., in the scalar curvature problem and the Yamabe problem) [27], nonlinear elasticity [19], non-Newtonian fluid mechanics [20], glaciology [34], mathematical biology [5], and elsewhere. As a result, questions concerning the solvability of problem (1.1) have received great attention, particularly after the famous work of Brezis and Nirenberg [13]. Among the vast number of results recorded in the literature so far, the case which has been studied extensively concerns the class of positive or non-negative solutions under a variety of the nonlinear term $g(x,u)$. However, an exhaustive review of the existing bibliography is beyond our present scope and the interested reader should consult the survey in [2], as well as the references cited therein.

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In recent years, problem (1.1), (1.2) have become rather an active area of research; see for example [6, 7, 14, 16, 17, 18, 22, 25, 26, 35, 37] and references therein. The existence of positive solutions of the problem in bounded domains is strongly dependent on a priori estimates of the solutions [29], so fewer results are known for \( N \geq 2 \). On the other hand, many papers deal with the existence of positive solutions of the problems in unbounded domains. For example, in [16, 18], the authors considered the existence and nonexistence of positive solutions of the problem (1.1) and (1.2) in upper half-space of \( \mathbb{R}^N \) with \( g(x,u) = -u + |u|^{p-2}u \) and \( f(x,u) = -|u|^{q-2}u \). They gave the exact form of the solution when \( p = \frac{2N}{N-2}, q = \frac{2(N-1)}{N-2} \) with \( N \geq 3 \) in [18], and proved the existence of positive solutions for \( p \geq \frac{2N}{N-2}, q \geq \frac{2(N-1)}{N-2} \) and the nonexistence of positive solutions for some cases of \( p \) and \( q \) in [16]. The nonexistence results of [16] in some sense can be regarded as an extension of the results in [26] where Hu considered the problem with \( g(x,u) = -u + a |u|^{p-2}u \) and \( f(x,u) = -|u|^{q-2}u \) in the exterior of a ball in the upper half-space of \( \mathbb{R}^N \). In [40], the author considers the multiplicity of positive solutions of the problem (1.1) and (1.2) in upper half-space of \( \mathbb{R}^N \) with \( g(x,u) = -u + |u|^{p-2}u \) and \( f(x,u) = b(x) |u|^{q-2}u \), with \( 1 < q < 2 < p < \frac{2N}{N-2} \) and \( b \) is a sign-changing continuous function. In [17] the authors proved that the number of sign-changing solutions strongly depends on the spatial dimension. For the existence and multiplicity of positive solutions by variational methods, see [6, 7, 14, 22, 25, 35, 37].

In this paper, we consider the existence and multiplicity of positive solutions for the following nonlinear boundary value problem:

\[
\begin{align*}
-\Delta u + u &= g_\lambda(x) |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\
\frac{\partial u}{\partial n} + f(x) |u|^{q-2}u &= 0 \quad \text{in } \partial\mathbb{R}^N_+,
\end{align*}
\tag{E_\lambda}
\]

where \( 1 < q < \min\{2_s, p\} \), \( 2_s = \frac{2(N-1)}{N-2} \) if \( N \geq 3 \), \( 2_s = \infty \) if \( N = 2 \), \( 2 < p < 2^* \) \( (2^* = \frac{2N}{N-2} \) if \( N \geq 3 \), \( 2^* = \infty \) if \( N = 2 \) \), \( \mathbb{R}^N_+ = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid x_N > 0\} \) is an upper half space in \( \mathbb{R}^N \), the parameter \( \lambda \in \mathbb{R} \) and \( g_\lambda(x) = 1 + \lambda a(x) \). We assume that the functions \( f \) and \( a \) satisfy the following conditions:

(D1) \( f \in C\left(\partial\mathbb{R}^N_+\right) \setminus \{0\} \) and there exists a positive number \( r_f > 1 \) such that

\[
0 \leq f(x) \leq \hat{c} \exp(-r_f |x|) \quad \text{for some } \hat{c} > 0 \text{ and for all } x \in \partial\mathbb{R}^N_+,
\]

(D2) \( a \in C\left(\overline{\mathbb{R}^N_+}\right) \) and there exists a positive number \( r_a \) such that

\[
a(x) \geq c_0 \exp(-r_a |x|) \quad \text{for some } c_0 > 0 \text{ and for all } x \in \overline{\mathbb{R}^N_+}
\]

and

\[
a(x) \to 0 \text{ uniformly as } |x| \to \infty.
\]
Theorem 1.1. Suppose that the functions $f$ and $a$ satisfy the conditions $(D1)$ and $(D2)$. Then there exists a positive number $\lambda_*$ such that equation $(E_\lambda)$ has at least three positive solutions for $\lambda \in (0, \lambda_*)$, and at least one positive solution for $\lambda \in \{0\} \cup [\lambda_*, \infty)$.

In the following sections, we proceed to prove Theorem 1.1. We use the variational methods to find positive solutions of equation $(E_\lambda)$. Associated with the equation $(E_\lambda)$, we consider the energy functional $J_\lambda$ in $H^1(\mathbb{R}_N^+)$

$$J_\lambda(u) = \frac{1}{2} \|u\|_{H^1}^2 + \frac{1}{q} \int_{\partial\mathbb{R}_N^+} f |u|^q d\sigma - \frac{1}{p} \int_{\mathbb{R}_N^+} g_\lambda |u|^p dx,$$

where $d\sigma$ is the measure on the boundary and $\|u\|_{H^1} = \left(\int_{\mathbb{R}_N^+} |\nabla u|^2 + u^2 dx\right)^{1/2}$ is the standard norm in $H^1(\mathbb{R}_N^+)$. It is well known that the solutions of equation $(E_\lambda)$ are the critical points of the energy functional $J_\lambda$ in $H^1(\mathbb{R}_N^+)$ (see Rabinowitz [36]).

This paper is organized as follows. In section 2, we give some notations and preliminaries. In section 3, we establish the existence of a positive solution for $\lambda = 0$. In section 4, we establish the existence of a positive solution for $\lambda > 0$. In sections 5, 6, we prove Theorem 1.1.

2. Notations and Preliminaries

First, we define the Palais–Smale (simply (PS)–) sequences, (PS)–values, and (PS)–conditions in $H^1(\mathbb{R}_N^+)$ for $J_\lambda$ as follows.

**Definition 2.1.** (i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(PS)_\beta$–sequence in $H^1(\mathbb{R}_N^+)$ for $J_\lambda$ if $J_\lambda(u_n) = \beta + o(1)$ and $J'_\lambda(u_n) = o(1)$ strongly in $H^{-1}(\mathbb{R}_N^+)$ as $n \to \infty$.

(ii) $J_\lambda$ satisfies the $(PS)_\beta$–condition in $H^1(\mathbb{R}_N^+)$ if every $(PS)_\beta$–sequence in $H^1(\mathbb{R}_N^+)$ for $J_\lambda$ contains a convergent subsequence.

As the energy functional $J_\lambda$ is not bounded below on $H^1(\mathbb{R}_N^+)$, it is useful to consider the functional on the Nehari manifold

$$N_\lambda = \{u \in H^1(\mathbb{R}_N^+) \setminus \{0\} \mid \langle J'_\lambda(u), u \rangle = 0\}.$$

Thus, $u \in N_\lambda$ if and only if

$$\|u\|_{H^1}^2 + \int_{\partial\mathbb{R}_N^+} f |u|^q d\sigma - \int_{\mathbb{R}_N^+} g_\lambda |u|^p dx = 0.$$

Then we have the following result.
Lemma 2.2. The energy functional $J_\lambda$ is coercive and bounded below on $N_\lambda$.

Proof. If $u \in N_\lambda$, then

$$J_\lambda(u) = \frac{p-2}{2p} \|u\|_{H^1}^2 + \frac{p-q}{pq} \int_{\partial \mathbb{R}_+^N} f |u|^q d\sigma \geq \frac{p-2}{2p} \|u\|_{H^1}^2.$$ (2.1)

Thus, $J_\lambda$ is coercive and bounded below on $N_\lambda$.

Define

$$\psi_\lambda(u) = \|u\|_{H^1}^2 + \int_{\partial \mathbb{R}_+^N} f |u|^q d\sigma - \int_{\mathbb{R}_+^N} g_\lambda |u|^p dx.$$

Then for $u \in N_\lambda$,

$$\langle \psi'_\lambda(u), u \rangle = 2 \|u\|_{H^1}^2 + q \int_{\partial \mathbb{R}_+^N} f |u|^q d\sigma - \int_{\mathbb{R}_+^N} g_\lambda |u|^p dx$$

$$= (2-p) \|u\|_{H^1}^2 + (q-p) \int_{\partial \mathbb{R}_+^N} f |u|^q d\sigma < 0.$$

Furthermore, we have the following result.

Lemma 2.3. Suppose that $u_0$ is a local minimizer for $J_\lambda$ on $N_\lambda$. Then $J'_\lambda(u_0) = 0$ in $H^{-1} (\mathbb{R}^N)$. Furthermore, if $u_0$ is a non-trivial nonnegative function in $\Omega$, then $u_0$ is a positive solution of equation $(E_\lambda)$.

Proof. By a similar argument to that in the proof of Brown and Zhang [15, Theorem 2.3] (or see Binding, Drábeč and Huang [8]), we have $J'_\lambda(u_0) = 0$ in $H^{-1} (\Omega)$, this implies that $u_0$ is a weak solution of equation $(E_\lambda)$. Now, if $u_0$ is a non-trivial nonnegative function in $\Omega$, then by the maximum principle, $u_0$ is positive in $\Omega$.

To get a better understanding of the Nehari manifold, we consider the function $m_u : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$m_u(t) = t^{2-q} \|u\|_{H^1}^2 - t^{p-q} \int_{\mathbb{R}_+^N} g_\lambda |u|^p dx \text{ for } t > 0.$$

Clearly, $tu \in N_\lambda$ if and only if $m_u(t) + \int_{\partial \mathbb{R}_+^N} f |u|^q d\sigma = 0$ and $m_u(\tilde{t}_\lambda(u)) = 0$, where

$$\tilde{t}_\lambda(u) = \left( \frac{\|u\|_{H^1}^2}{\int_{\mathbb{R}_+^N} g_\lambda |u|^p dx} \right)^{\frac{1}{p-2}} > 0.$$

Moreover,

$$m'_u(t) = t^{1-q} \left[ (2-q) \|u\|_{H^1}^2 - (p-q)t^{p-2} \int_{\mathbb{R}_+^N} g_\lambda |u|^p dx \right].$$
Thus, if \( q \geq 2 \), then
\[
m'_u(t) < 0 \text{ for all } t > 0,
\]
which implies that \( m_u \) is strictly decreasing on \((0, \infty)\) with \( \lim_{t \to 0^+} m_u(t) = \infty \) and \( \lim_{t \to \infty} m_u(t) = -\infty \), and if \( q < 2 \), then \( m_u \) has a unique critical point at \( t = \tilde{t}_\lambda(u) < \hat{t}_\lambda(u) \), where
\[
\tilde{t}_\lambda(u) = \left( \frac{(2-q)\|u\|_{H^1}^2}{(p-q) \int_{\mathbb{R}_+^N} g_\lambda |u|^p \, dx} \right)^{\frac{1}{p-2}} > 0
\]
such that
\[
m_u(\tilde{t}_\lambda(u)) = \|u\|_{H^1}^q \left( \frac{p-2}{p-q} \right) \left( \frac{2-q}{p-q} \right)^{\frac{2-q}{p-2}} \left( \frac{\|u\|_{H^1}^p}{\int_{\mathbb{R}_+^N} g_\lambda |u|^p \, dx} \right)^{\frac{2-q}{p-2}} > 0,
\]
which implies that \( m_u \) is strictly increasing on \((0, \tilde{t}_\lambda(u))\) and \( m_u \) is strictly decreasing on \((\tilde{t}_\lambda(u), \infty)\) with \( \lim_{t \to \infty} m_u(t) = -\infty \). Therefore, we can conclude that for each \( u \in H^1(\mathbb{R}_+^N) \setminus \{0\} \) there exist \( 0 < \tilde{t}_\lambda(u) < \hat{t}_\lambda(u) \) such that \( m_u(t) > 0 \) for all \( t \in (0, \tilde{t}_\lambda(u)) \) and \( m_u \) is strictly decreasing on \([\tilde{t}_\lambda(u), \infty)\) with \( m_u(t) < 0 \) for all \( t \in ([\tilde{t}_\lambda(u), \infty) \). Moreover, we have the following lemma.

**Lemma 2.4.** Suppose that \( \lambda \geq 0 \). Then for each \( u \in H^1(\mathbb{R}_+^N) \setminus \{0\} \) we have the following.

(i) There is a unique \( t_\lambda(u) \geq \tilde{t}_\lambda(u) \) such that \( t_\lambda(u) u \in N_\lambda \). Furthermore,
\[
J_\lambda(t_\lambda(u) u) = \sup_{t \geq 0} J_\lambda(t u) = \sup_{t \geq \tilde{t}_\lambda(u)} J_\lambda(t u). \tag{2.2}
\]

(ii) \( t_\lambda(u) \) is a continuous function for \( u \in H^1(\mathbb{R}_+^N) \setminus \{0\} \).

(iii) \( t_\lambda(u) = \frac{1}{\|u\|_{H^1}^2} \lambda \left( \frac{u}{\|u\|_{H^1}^2} \right) \).

(iv) \( N_\lambda = \left\{ u \in H^1(\mathbb{R}_+^N) \setminus \{0\} \mid \frac{1}{\|u\|_{H^1}^2} \lambda \left( \frac{u}{\|u\|_{H^1}^2} \right) = 1 \right\} \).

**Proof.** Fix \( u \in H^1(\mathbb{R}_+^N) \setminus \{0\} \).

(i) Let
\[
h_u(t) = J_\lambda(tu) = \frac{t^2}{2} \|u\|_{H^1}^2 + \frac{t^q}{q} \int_{\partial \mathbb{R}_+^N} f |u|^q \, d\sigma - \frac{t^p}{p} \int_{\mathbb{R}_+^N} g_\lambda |u|^p \, dx.
\]
Then
\[
h_u'(t) = \frac{t^q}{q} \left( m_u(t) + \int_{\partial \mathbb{R}_+^N} f |u|^q \, d\sigma \right).
\]
Thus, by \( \int_{\partial \mathbb{R}^N} f \, |u|^q \, d\sigma \geq 0 \), the equation \( m_u(t) + \int_{\partial \mathbb{R}^N} f \, |u|^q \, d\sigma = 0 \) has a unique solution \( t_\lambda(u) \geq \hat{t}_\lambda(u) \), which implies that \( h'_u(t\lambda(u)) = 0 \) and \( t\lambda(u) \in \mathbf{N}_\lambda \). Moreover, \( h_u \) is strictly increasing on \((0, t\lambda(u))\) and strictly decreasing on \((t\lambda(u), \infty)\). Therefore, \( (2.2) \) holds.

(ii) By the uniqueness of \( t\lambda(u) \) and the extremal property of \( t\lambda(u) \), we have \( t\lambda(u) \) is a continuous function for \( u \in H^1(\mathbb{R}_+^N) \setminus \{0\} \).

(iii) Let \( v = \frac{u}{\|u\|_{H^1}} \). Then, by part (i), there is a unique \( t\lambda(v) > 0 \) such that \( t\lambda(v) \in \mathbf{N}_\lambda \) or \( t\lambda\left(\frac{u}{\|u\|_{H^1}}\right) \frac{u}{\|u\|_{H^1}} \in \mathbf{N}_\lambda \). Thus, by the uniqueness of \( t\lambda(v) \), we can conclude that \( t\lambda(u) = \frac{1}{\|u\|_{H^1}} \hat{t}_\lambda\left(\frac{u}{\|u\|_{H^1}}\right) \).

(iv) For \( u \in \mathbf{N}_\lambda \). By parts (i), (iii), \( t\lambda\left(\frac{u}{\|u\|_{H^1}}\right) \frac{u}{\|u\|_{H^1}} \in \mathbf{N}_\lambda \). Since \( u \in \mathbf{N}_\lambda \), we have \( t\lambda\left(\frac{u}{\|u\|_{H^1}}\right) \frac{u}{\|u\|_{H^1}} = 1 \), which implies that

\[
\mathbf{N}_\lambda \subset \left\{ u \in H^1(\mathbb{R}_+^N) \mid \frac{1}{\|u\|_{H^1}} t\lambda\left(\frac{u}{\|u\|_{H^1}}\right) = 1 \right\}.
\]

Conversely, let \( u \in H^1(\mathbb{R}_+^N) \) such that \( \frac{1}{\|u\|_{H^1}} t\lambda\left(\frac{u}{\|u\|_{H^1}}\right) = 1 \). Then, by part (iii),

\[
t\lambda\left(\frac{u}{\|u\|_{H^1}}\right) \frac{u}{\|u\|_{H^1}} \in \mathbf{N}_\lambda.
\]

Thus,

\[
\mathbf{N}_\lambda = \left\{ u \in H^1(\mathbb{R}_+^N) \setminus \{0\} \mid \frac{1}{\|u\|_{H^1}} t\lambda\left(\frac{u}{\|u\|_{H^1}}\right) = 1 \right\}.
\]

This completes the proof.

Now we consider the following elliptic problems:

\[
\begin{aligned}
&\begin{cases}
-\Delta u + u = |u|^{p-2} u & \text{in } \mathbb{R}_+^N, \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \mathbb{R}_+^N.
\end{cases} \\
&\text{(\(E^\infty\))}
\end{aligned}
\]

and

\[
\begin{aligned}
&\begin{cases}
-\Delta u + u = |u|^{p-2} u & \text{in } \mathbb{R}^N, \\
\lim_{|x| \to \infty} u = 0.
\end{cases} \\
&\text{(\(\tilde{E}^\infty\))}
\end{aligned}
\]

Associated with the equations \((E^\infty)\) and \((\tilde{E}^\infty)\), we consider the energy functionals \(J^\infty\) in \(H^1(\mathbb{R}_+^N)\) and \(\tilde{J}^\infty\) in \(H^1(\mathbb{R}^N)\)

\[
J^\infty(u) = \frac{1}{2} \int_{\mathbb{R}_+^N} |\nabla u|^2 + u^2 \, dx - \frac{1}{p} \int_{\mathbb{R}_+^N} |u|^p \, dx
\]

and

\[
\tilde{J}^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \, dx,
\]
respectively. Consider the minimizing problems:

$$
\inf_{u \in \mathcal{N}^\infty} J^\infty (u) = \alpha^\infty \text{ and } \inf_{u \in \mathcal{N}^\infty} \tilde{J}^\infty (u) = \tilde{\alpha}^\infty,
$$

where

$$
\mathcal{N}^\infty = \{ u \in H^1 (\mathbb{R}_+^N) \setminus \{ 0 \} \mid \langle (J^\infty)' (u), u \rangle = 0 \}
$$

and

$$
\tilde{\mathcal{N}}^\infty = \{ u \in H^1 (\mathbb{R}^N) \setminus \{ 0 \} \mid \langle (\tilde{J}^\infty)' (u), u \rangle = 0 \}.
$$

It is known that equations \((E^\infty)\) and \((\tilde{E}^\infty)\) have unique positive radial solutions \(w(x)\) and \(\tilde{w}(x)\), respectively such that \(J^\infty (w) = \alpha^\infty\) and \(\tilde{J}^\infty (\tilde{w}) = \tilde{\alpha}^\infty\) (see [21, 28, 32, 33]). Without loss of generality, we may assume that

$$
w(0) = \max_{x \in \partial \mathbb{R}_+^N} w(x) = \max_{x \in \mathbb{R}_+^N} w(x)
$$

and

$$
\tilde{w}(0) = \max_{x \in \partial \mathbb{R}_+^N} \tilde{w}(x) = \max_{x \in \mathbb{R}^N} \tilde{w}(x).
$$

Thus, we observe that solution \(w(x)\) can construct solution \(\tilde{w}(x)\) of equation \((\tilde{E}^\infty)\) by reflection with respect to \(\partial \mathbb{R}_+^N\). Then \(\tilde{\alpha}^\infty = \tilde{J}^\infty (\tilde{w}) = 2J^\infty (w) = 2\alpha^\infty\) (or see [21, p. 889]). For \(\lambda \geq 0\), similarly as in [10, 11, 30, 31], we have the following results.

**Proposition 2.5.** Let \(\{ u_n \}\) be a \((PS)_\delta\)-sequence in \(H^1 (\mathbb{R}_+^N)\) for \(J_\lambda\). Then there exist a subsequence \(\{ u_n \}\), \(m, \tilde{m} \in \mathbb{N} \cup \{ 0 \}\), sequences \(\{ x^i_n \} \subset \partial \mathbb{R}_+^N\) and \(\{ \tilde{x}^j_n \} \subset \mathbb{R}^N\), function \(v_0 \in H^1 (\mathbb{R}_+^N)\), \(0 \neq w^i \in H^1 (\mathbb{R}_+^N)\), for \(1 \leq i \leq m\) and \(0 \neq \tilde{w}^j \in H^1 (\mathbb{R}^N)\), for \(1 \leq j \leq \tilde{m}\) such that

\(\text{(i)}\) \(|x^i_n| \to \infty\) and \(|x^i_n - x^k_n| \to \infty\) as \(n \to \infty\), for \(1 \leq i \neq k \leq m\);

\(\text{(ii)}\) \(|\tilde{x}^j_n| \to \infty\) and \(|\tilde{x}^j_n - \tilde{x}^k_n| \to \infty\) as \(n \to \infty\), for \(1 \leq j \neq k \leq \tilde{m}\);

\(\text{(iii)}\) \(|x^i_n - \tilde{x}^j_n| \to \infty\) as \(n \to \infty\), for \(1 \leq i \leq m\) and \(1 \leq j \leq \tilde{m}\);

\(\text{(iv)}\)

\[
\begin{align*}
-\Delta v_0 + v_0 &= g_\lambda (x) |v_0|^{p-2} v_0 \quad \text{in } \mathbb{R}_+^N, \\
\frac{\partial v_0}{\partial n} + f (x) |v_0|^{q-2} v_0 &= 0 \quad \text{on } \partial \mathbb{R}_+^N,
\end{align*}
\]

\(\text{(v)}\)

\[
\begin{align*}
-\Delta w^i + w^i &= |w^i|^{p-2} w^i \quad \text{in } \mathbb{R}_+^N, \\
\frac{\partial w^i}{\partial n} &= 0 \quad \text{on } \partial \mathbb{R}_+^N,
\end{align*}
\]

\(\text{(vi)}\)

\[
-\Delta \tilde{w}^j + \tilde{w}^j = |\tilde{w}^j|^{p-2} \tilde{w}^j \quad \text{in } \mathbb{R}^N;
\]
(vii) \( u_n = v_0 + \sum_{i=1}^{m} w^i (\cdot - x^i_n) + \tilde{m} \tilde{w}^j (\cdot - \tilde{x}^j_n) + o(1) \) strongly in \( H^1 (\mathbb{R}^N) \);

(viii) \( \|u_n\|_{H^1 (\mathbb{R}^N)}^2 = \|v_0\|_{H^1 (\mathbb{R}^N)}^2 + \sum_{i=1}^{m} \|w^i\|_{H^1 (\mathbb{R}^N)}^2 + \tilde{m} \|\tilde{w}^j\|_{H^1 (\mathbb{R}^N)}^2 + o(1) \);

(viii) \( J_\lambda(u_n) = J_\lambda(v_0) + \sum_{i=1}^{m} J^\infty(w^i) + \tilde{m} J^\infty(\tilde{w}^j) + o(1) \);

In addition, if \( u_n \geq 0 \), then \( v_0 \geq 0, w^i \geq 0 \) and \( \tilde{w}^j \geq 0 \) for each \( 1 \leq i \leq m \) and \( 1 \leq j \leq \tilde{m} \).

**Proof.** Since \( \{u_n\} \) is a (PS)\(_\beta\)-sequence in \( H^1 (\mathbb{R}^N_+) \) for \( J_\lambda \), by Lemma 2.2, there is a subsequence \( \{u_n\} \) and \( v_0 \) in \( H^1 (\mathbb{R}^N_+) \) such that

\[ u_n \rightharpoonup v_0 \text{ weakly in } H^1 (\mathbb{R}^N_+) \text{ and in } L^p (\mathbb{R}^N_+) \quad (2.3) \]

and \( v_0 \) is a solution of equation \((E_\lambda)\). Let \( \tilde{u}_n = u_n - v_0 \). Then, by (2.3),

\[ \tilde{u}_n \rightharpoonup 0 \text{ weakly in } H^1 (\mathbb{R}^N_+) \text{ and in } L^p (\mathbb{R}^N_+) . \]

Suppose that \( \tilde{u}_n \not\rightarrow 0 \) strongly in \( H^1 (\mathbb{R}^N_+) \) (otherwise, the result is automatic). By a similar argument to that in the proof of [11, Lemma 3.1], there exist \( \delta > 0 \) and \( \{z_n\} \in \mathbb{R}^{N-1} \times \mathbb{R} \) such that \( |x_n| \rightarrow \infty \) and

\[ \int_{B^N (1) + x_n} |\tilde{u}_n|^2 dx > \delta , \]

where \( B^N (1) = \{ x \in \mathbb{R}^N \mid |x| < 1 \} \). Moreover, we may assume that one of the following two cases occurs:

(a) \( \{z_n\} \) is bounded;

(b) \( \{z_n\} \) is unbounded.

Case (a) : Without loss of generality, we may assume that \( z_n = 0 \). Set \( x^i_n = x_n = (y_n, 0) \). Then \( |x_n^i| \rightarrow \infty \) and \( \tilde{u}_n (x + x^i_n) \in H^1 (\mathbb{R}^N_+) \) for all \( n \). Adopting the approach employed in the proof of [10, Proposition II.1] (or see [22, Proposition 2.1]), there exists \( w^1 \in H^1 (\mathbb{R}^N_+) \setminus \{0\} \) such that

\[ \tilde{u}_n (x + x^1_n) \rightharpoonup w^1 \text{ weakly in } H^1 (\mathbb{R}^N_+) \]

and

\[ \left\{ \begin{array}{ll}
- \Delta w^1 + w^1 = |w^1|^{p-2} w^1 \quad & \text{in } \mathbb{R}^N_+ , \\
\frac{\partial w^1}{\partial n} = 0 & \text{on } \partial \mathbb{R}^N_+ .
\end{array} \right. \]

Case (b) : Set \( \tilde{x}^1_n = x_n = (y_n, z_n) \). Then \( |\tilde{x}^1_n| \rightarrow \infty \) and \( \tilde{u}_n (x + \tilde{x}^1_n) \in H^1 (\mathbb{R}^N) \) for all \( n \). Again, using a similar procedure to that in the proof of [10, Proposition II.1] (or see [11, Lemma 3.1]), there exists \( \tilde{w}^1 \in H^1 (\mathbb{R}^N) \setminus \{0\} \) such that

\[ \tilde{u}_n (x + x^1_n) \rightharpoonup w^1 \text{ weakly in } H^1 (\mathbb{R}^N) \]
and \[-\Delta \tilde{w}^1 + \tilde{w}^1 = |\tilde{w}^1|^{p-2} \tilde{w}^1 \text{ in } \mathbb{R}^N.\]

Following the same lines of the proof in [10, Proposition II.1] (or see [11, Lemma 3.1]), we repeat the argument above, each iteration will likewise give rise to two cases and the procedure will terminate after some finite steps; the procedure will also lead us to conclude that \((i)-(viiii)\) hold.

For \(\lambda \geq 0\), we define
\[
\alpha_\lambda = \inf_{u \in \mathbb{N}_\lambda} J_\lambda(u).
\]

Then, by Proposition 2.5, we have the following compactness result.

**Corollary 2.6.** Suppose that \(\{u_n\}\) is a \((PS)_\beta\)-sequence in \(H^1(\mathbb{R}_+^N)\) for \(J_\lambda\) with \(0 < \beta < \alpha^\infty + \min \{\alpha^\infty, \alpha_\lambda\}\) and \(\beta \neq \alpha^\infty\). Then there exists a subsequence \(\{u_n\}\) and a non-zero \(v_0\) in \(H^1(\mathbb{R}_+^N)\) such that \(u_n \to v_0\) strongly in \(H^1(\mathbb{R}_+^N)\) and \(J_\lambda(v_0) = \beta\). Furthermore, \(v_0\) is a non-zero solution of equation \((E_\lambda)\).

**Proof.** Since \(\tilde{\alpha}^\infty = 2\alpha^\infty\) and
\[
\beta < \alpha^\infty + \min \{\alpha^\infty, \alpha_\lambda\} \leq 2\alpha^\infty,
\]
we have \(\tilde{m} = 0\), which implies that
\[
\|u_n\|_{H^1(\mathbb{R}_+^N)}^2 = \|v_0\|_{H^1(\mathbb{R}_+^N)}^2 + \sum_{i=1}^m \|w^i\|_{H^1(\mathbb{R}_+^N)}^2 + o(1)
\]
and
\[
\beta = J_\lambda(v_0) + \sum_{i=1}^m J^\infty(w^i).
\]

Since \(0 < \beta < \alpha^\infty + \min \{\alpha^\infty, \alpha_\lambda\}\) and \(\beta \neq \alpha^\infty\), by the uniqueness of positive solutions of equation \((E^\infty)\), we conclude that \(m = 0\). Thus, \(u_n \to v_0\) strongly in \(H^1(\mathbb{R}_+^N)\) and \(J_\lambda(v_0) = \beta > 0\), which implies that \(v_0\) is a non-zero solution of equation \((E_\lambda)\).

3. **Existence of positive solutions for \(\lambda = 0\)**

Let \(w(x)\) be a positive radial solution of equation \((E^\infty)\) such that \(J^\infty(w) = \alpha^\infty\). Then, by Gidas, Ni and Nirenberg [24] and Kwong [28], for any \(\varepsilon > 0\), there exist positive numbers \(A_\varepsilon\) and \(B_0\) such that
\[
A_\varepsilon \exp \left(-1 + \varepsilon \right) \leq w(x) \leq B_0 \exp (-|x|) \text{ for all } x \in \mathbb{R}_+^N.
\]

Let \(y \in \mathbb{S} = \{x' \in \mathbb{R}^{N-1} \mid |x'| = 1\}\) and let \(z_0 = (\delta_0, 0, \ldots, 0) \in \mathbb{R}^{N-1}\), where
\[
0 < \delta_0 = \frac{\min \{r_f, q, \frac{p}{2}\} - 1}{2 \left( \min \{r_f, q, \frac{p}{2}\} + 1 \right)} < 1.
\]
Clearly,
\[1 - \delta_0 \leq |y - z_0| \leq 1 + \delta_0 \text{ for all } y \in S,\] (3.2)

Define
\[w_{y,l}(x) = w(x - l(y, 0)) \text{ for } l \geq 0 \text{ and } y \in S\] (3.3)
and
\[w_{z_0,l}(x) = w(x - l(z_0, 0)) \text{ for } l \geq 0.\]

Clearly, \(w_{y,l}\) and \(w_{z_0,l}\) are also least energy positive solutions of equation \((E^\infty)\) for all \(l \geq 0\). Moreover, by Lemma 2.4 for each \(u \in H^1(\mathbb{R}^N_+) \setminus \{0\}\) there is a unique \(t_0(u) \geq 0\) such that \(t_0(u) u \in N_0\). Then we have the following results.

**Lemma 3.1.** For each \(s_0 \in (0, 1)\) there exist \(l(s_0) > 0\) and \(\sigma(s_0) > 1\) such that for any \(l > l(s_0)\), we have
\[\hat{\tau}_0^{p-2} (sw_{y,l} + (1 - s) w_{z_0,l}) > \sigma(s_0) \frac{\sigma(s_0)}{s^{p-2} + (1 - s)^{p-2}}\]
for all \(y \in S\) and for all \(s \in (0, 1)\) with \(\min\{s, 1 - s\} \geq s_0\).

**Proof.** Since
\[
\hat{\tau}_0^{p-2} (sw_{y,l} + (1 - s) w_{z_0,l}) = \frac{\|sw_{y,l} + (1 - s) w_{z_0,l}\|_{H^1}^2}{\int_{\mathbb{R}^N_+} |sw_{y,l} + (1 - s) w_{z_0,l}|^p dx} \\
= \frac{s^2 \|w\|_{H^1}^2 + (1 - s)^2 \|w\|_{H^1}^2 + 2s(1 - s) \langle w_{y,l}, w_{z_0,l} \rangle}{\int_{\mathbb{R}^N} |sw_{y,z_0,l} + (1 - s) w|^p dx} \] (3.4)
for all \(s \in [0, 1]\) and for all \(y \in S\). Moreover, by (3.1), the triangle inequality and
\[1 - \delta_0 \leq |y - z_0| \leq 1 + \delta_0 \text{ for all } y \in S,\] (3.5)
we have
\[
\langle w_{y,l}, w_{z_0,l} \rangle = \int_{\mathbb{R}_+^N} w^{p-1} w_{y-z_0,l} \, dx
\]
\[
\leq B_0^p \int_{|x| < (1 + \delta_0)l} \exp \left(- (|x| + |x - l(z_0 - y,0)|) \right) \, dx
\]
\[+ B_0^p \int_{|x| \geq (1 + \delta_0)l} \exp \left(- (|x| + |x - l(z_0 - y,0)|) \right) \, dx
\]
\[= B_0^p l^N \int_{|x| < (1 + \delta_0)} \exp \left(- l(|x| + |x - (z_0 - y,0)|) \right) \, dx
\]
\[+ B_0^p \int_{|x| \geq (1 + \delta_0)l} \exp \left(- |x| \right) \exp \left(- |x - l(z_0 - y,0)| \right) \, dx
\]
\[
\leq B_0^p l^N \int_{|x| < (1 + \delta_0)} \exp \left(- l(|x| + |x - (z_0 - y,0)|) \right) \, dx
\]
\[+ B_0^p \exp \left(- (1 + \delta_0)l \right) l \int_{|x| \geq (1 + \delta_0)l} \exp \left(- (|x - l(z_0 - y,0)|) \right) \, dx
\]
\[
\leq B_0^p l^N \int_{|x| < (1 + \delta_0)} \exp \left(- l(|y - z_0|) \right) \, dx
\]
\[+ B_0^p \exp \left(- (1 + \delta_0)l \right) \int_{\mathbb{R}_+^N} \exp \left(- |x| \right) \, dx
\]
\[
\leq B_0^p l^N \exp \left(- (1 - \delta_0)l \right) l \int_{|x| < (1 + \delta_0)} 1 \, dx + d_0 B_0^p \exp \left(- (1 + \delta_0)l \right)
\]
\[
\leq C_0 B_0^p l^N \exp \left(- l(1 - \delta_0) \right) \quad \text{for all } l \geq 1 \text{ and for all } y \in \mathbb{S},
\]
which implies that
\[
\lim_{l \to \infty} \langle w_{y,l}, w_{z_0,l} \rangle = 0 \quad \text{uniformly in } y \in \mathbb{S}. \quad (3.6)
\]
By (3.1), (3.5) and Brézis-Lieb lemma [12], for any \( s \in [0,1] \) we have
\[
\lim_{l \to \infty} \int_{\mathbb{R}_+^N} |s w_{y,z_0,l} + (1 - s) w|^p - |s w_{y,z_0,l}|^p \, dx
\]
\[= \int_{\mathbb{R}_+^N} |(1 - s) w|^p \, dx \quad \text{uniformly in } y \in \mathbb{S}. \quad (3.7)
\]
Thus, by (3.4), (3.6) and (3.7), for any \( s \in [0,1] \)
\[
\lim_{l \to \infty} \frac{B_0^p - (s w_{y,l} + (1 - s) w_{z_0,l})}{||w||_{H^1}} = \frac{(s^2 + (1 - s)^2) ||w||_{H^1}^2}{(s^p + (1 - s)^p) \int_{\mathbb{R}^N} |w|^p \, dx}
\]
\[= \frac{s^2 + (1 - s)^2}{s^p + (1 - s)^p} \quad \text{uniformly in } y \in \mathbb{S}. \quad (3.8)
\]
Since
\[
\frac{(s^2 + (1-s)^2)(s^{p-2} + (1-s)^{p-2})}{s^p + (1-s)^p} = 1 + \frac{s^2(1-s)^{p-2} + (1-s)^2s^{p-2}}{s^p + (1-s)^p} \\
\geq 1 + \frac{s_0^2(1-s_0)^{p-2} + (1-s_0)^2s_0^{p-2}}{s_0^p + (1-s_0)^p}
\]
for all \(s \in (0,1)\) with \(\min \{s,1-s\} > s_0\), by (3.8) and (3.9), there exist \(l(s_0) > 0\) and \(\sigma(s_0) > 1\) such that for any \(l > l(s_0)\), we have
\[
\tilde{t}^{p-2}_0(sw_{y,l} + (1-s)w_{z_0,l}) > \frac{\sigma(s_0)}{s_0^{p-2} + (1-s)^{p-2}}
\]
for all \(y \in S\) and for all \(s \in (0,1)\) with \(\min \{s,1-s\} \geq s_0\). This completes the proof.

**Proposition 3.2.** There exists \(l_1 > 0\) such that for any \(l \geq l_1\)
\[
\sup_{t \geq 0} J_0(t[sw_{y,l} + (1-s)w_{z_0,l}]) < 2\alpha^\infty \text{ for all } y \in S,
\]
where \(J_0 = J_\lambda\) for \(\lambda = 0\). Furthermore, there is a unique \(t_0(sw_{y,l} + (1-s)w_{z_0,l}) > 0\) such that
\[
t_0(sw_{y,l} + (1-s)w_{z_0,l})[sw_{y,l} + (1-s)w_{z_0,l}] \in N_0.
\]

**Proof.** When \(s = 0\) or \(1\), by a similar argument to that in the proof of Wu [40, Proposition 2], there exists \(\tilde{t}_1 > 0\) such that
\[
\max \left\{ \sup_{t \geq 0} J_0(tw_{y,l}), \sup_{t \geq 0} J_0(tw_{z_0,l}) \right\} \leq \alpha^\infty + \frac{\tilde{t}_1 C_0}{q} \exp \left( - \min \{r_f, q \} l \right)
\]
for all \(y \in S\), this implies that there exists \(\tilde{t}_1 > 0\) such that for any \(l > \tilde{t}_1\),
\[
\max \left\{ \sup_{t \geq 0} J_0(tw_{y,l}), \sup_{t \geq 0} J_0(tw_{z_0,l}) \right\} \leq \frac{3}{2} \alpha^\infty \text{ for all } y \in S.
\]
Therefore, by \(J_0 \in C^2(H^1(\mathbb{R}^N_+, \mathbb{R})\) and (3.11), there exist positive constants \(s_0, \tilde{l}\) such that for any \(l > \tilde{l}\),
\[
\sup_{t \geq 0} J_0(t[sw_{y,l} + (1-s)w_{z_0,l}]) < 2\alpha^\infty
\]
for all \( y \in S \) and for all \( \min \{ s, 1 - s \} \leq s_0 \). In the following we always assume that \( \min \{ s, 1 - s \} \geq s_0 \). Since

\[
J_0(t \left[ s w_{y,l} + (1 - s) w_{z_0,l} \right])
\]

\[
= \frac{t^2}{2} \left[ s^2 \| w \|_{H^1}^2 + (1 - s)^2 \| w \|_{H^1}^2 + 2 s (1 - s) \langle w_{y,l}, w_{z_0,l} \rangle \right]
\]

\[
+ \frac{t^q}{q} \int_{\partial \mathbb{R}^N_+} f (s w_{y,l} + (1 - s) w_{z_0,l})^q \, d\sigma
\]

\[
- \frac{t^p}{p} \int_{\mathbb{R}^N_+} (s w_{y,l} + (1 - s) w_{z_0,l})^p \, dx
\]

\[
\leq \frac{t^2}{2} \left[ s^2 + 2 s (1 - s) + (1 - s)^2 \right] \| w \|_{H^1}^2
\]

\[
+ \frac{\tilde{c} t^{q-1}}{q} \left( s^q + (1 - s)^q \right) \int_{\partial \mathbb{R}^N_+} w^q \, d\sigma - \frac{t^p}{p} \max \left\{ s^p, (1 - s)^p \right\} \int_{\mathbb{R}^N_+} w^p \, dx
\]

\[
\leq t^2 \| w \|_{H^1}^2 + \frac{\tilde{c} t^{q-1}}{q} \int_{\partial \mathbb{R}^N_+} w^q \, d\sigma - \frac{t^p}{p^2} \int_{\mathbb{R}^N_+} w^p \, dx
\]

(3.13)

for all \( 0 \leq s \leq 1 \) and \( y \in S \). Then there exists \( t_1 > 0 \) such that for any \( t \geq t_1 \),

\[
J_0(t \left[ s w_{y,l} + (1 - s) w_{z_0,l} \right]) < 2 \alpha^\infty \text{ for all } 0 \leq s \leq 1 \text{ and for all } y \in S. \quad (3.14)
\]

Moreover, by Lemma 2.4 (i) and Lemma 3.1,

\[
\sup_{t \geq 0} J_0(t \left[ s w_{y,l} + (1 - s) w_{z_0,l} \right]) = \sup_{t \geq t_2} J_0(t \left[ s w_{y,l} + (1 - s) w_{z_0,l} \right]) \quad (3.15)
\]

for all \( y \in S \), where \( t_2 = \left( \frac{\sigma(s_0)}{s^{p-2} + (1 - s)^{p-2}} \right)^{1/(p-2)} \) and \( \sigma(s_0) > 1 \) is as in Lemma 3.1. Thus, by (3.14) and (3.15), we only need to show that there exists \( l_1 \geq \tilde{l} \) such that for any \( l > l_1 \),

\[
\sup_{t_2 \leq t \leq t_1} J_0(t \left[ s w_{y,l} + (1 - s) w_{z_0,l} \right]) < 2 \alpha^\infty \text{ for all } y \in S. \quad (3.16)
\]

By lemma 2.1 in Bahri and Li [9], there exists \( C_p > 0 \), such that for any nonnegative real numbers \( a, b \),

\[
(a + b)^p \geq a^p + b^p + p \left( a^{p-1} b + a b^{p-1} \right) - C_p a^{p/2} b^{p/2}. \quad (3.17)
\]
Then, by (3.13), (3.17) and Lemma 3.1,
\[
J_0(t [sw_{y,l} + (1 - s) w_{z_0,l}])
\leq 2\alpha_\infty + s (1 - s) \left[ t^2 - t^p s^{p-2} - t^p (1 - s)^{p-2} \right] \int_{\mathbb{R}^N_+} w^{p-1}_{y,l} w_{z_0,l} dx
\]
\[
+ \frac{t^q}{q} \int_{\partial\mathbb{R}^N_+} f (sw_{y,l} + (1 - s) w_{z_0,l})^q d\sigma + \frac{t^p C_p}{p} \int_{\mathbb{R}^N_+} w^{p/2}_{y,l} w^{p/2}_{z_0,l} dx
\]
\leq 2\alpha_\infty - C_0^2 \sigma (s_0) - 1 \int_{\mathbb{R}^N_+} w^{p-1}_{y,l} w_{z_0,l} dx
\]
\[
+ \frac{t^q}{q} \int_{\partial\mathbb{R}^N_+} f (sw_{y,l} + (1 - s) w_{z_0,l})^q d\sigma + \frac{t^p C_p}{p} \int_{\mathbb{R}^N_+} w^{p/2}_{y,l} w^{p/2}_{z_0,l} dx
\]
for all \( y \in S \), where we have used the result
\[
\int_{\mathbb{R}^N_+} w^{p-1}_{y,l} w_{z_0,l} dx = \langle w_{y,l}, w_{z_0,l} \rangle = \int_{\mathbb{R}^N_+} w_{y,l} w^{p-1}_{z_0,l} dx.
\]

We first estimate \( \int_{\mathbb{R}^N_+} w^{p-1}_{y,l} w_{z_0,l} dx \). Setting
\[
\mathcal{C}_0 = \min_{x \in B^N_+(e_N, \frac{1}{2})} w^{p-1} (x) > 0,
\]
where \( B^N_+(e_N, \frac{1}{2}) = \{ x \in \mathbb{R}^N_+ | |x - e_N| \leq \frac{1}{2} \} \) and \( e_N = (0, \ldots, 0, 1) \in \mathbb{R}^N \). Then, by (3.1) and (3.2), for any \( \varepsilon > 0 \),
\[
\int_{\mathbb{R}^N_+} w^{p-1}_{y,l} w_{z_0,l} dx = \int_{\mathbb{R}^N_+} w^{p-1} (x) w (x - l (z_0 - y, 0)) dx
\]
\[
\geq \mathcal{C}_0 A_x \int_{B^N_+(e_N, \frac{1}{2})} \exp \left( - (1 + \varepsilon) |x - l (z_0 - y, 0)| \right) dx
\]
\[
\geq \mathcal{C}_0 A_x \int_{B^N_+(e_N, \frac{1}{2})} \exp \left( - (1 + \varepsilon) |x| - l (1 + \varepsilon) |y - z_0| \right) dx
\]
\[
\geq \mathcal{C}_0 A_x \exp \left( - l (1 + \varepsilon) |y - z_0| \right)
\]
\[
\geq \mathcal{C}_0 A_x \exp \left( - l (1 + \varepsilon) (1 + \delta_0) \right). \tag{3.19}
\]
From (3.2) we have

\[
\int_{\mathbb{R}^N_+} w_{y,l}^{p/2} w_{z_0,l}^{p/2} \, dx \\
\leq B_0^p \int_{|x|<(1+\delta_0)l} \exp \left( -\frac{p}{2} (|x| + |x - l (z_0 - y, 0)|) \right) \, dx \\
+ B_0^p \int_{|x|\geq(1+\delta_0)l} \exp \left( -\frac{p}{2} (|x| + |x - l (z_0 - y, 0)|) \right) \, dx \\
\leq B_0^{plN} \int_{|x|<(1+\delta_0)l} \exp \left( -\frac{p}{2} l (|x| + |x - (z_0 - y, 0)|) \right) \, dx \\
+ B_0^p \exp \left( -\frac{(1+\delta_0)pl}{2} \right) \int_{|x|\geq(1+\delta_0)l} \exp \left( -\frac{p}{2} (|x - l (y - z_0, 0)|) \right) \, dx \\
\leq B_0^{plN} \int_{|x|<(1+\delta_0)l} \exp \left( -\frac{pl}{2} |y - z_0| \right) \, dx + \hat{C}_0 B_0^p \exp \left( -\frac{pl}{2} |y - z_0| \right) \\
\leq C_0 B_0^{plN} \exp \left( -\min \left\{ r_f, q, \frac{p}{2} \right\} (1 - \delta_0) l \right) \text{ for } l \geq 1. 
\tag{3.20}
\]

By (D2), we also have

\[
\int_{\partial\mathbb{R}^N_+} f (sw_{y,l} + (1 - s) w_{z_0,l})^q \, d\sigma \\
\leq 2^{q-1} \left( \int_{\partial\mathbb{R}^N_+} f w_{y,l}^q \, d\sigma + \int_{\partial\mathbb{R}^N_+} f w_{z_0,l}^q \, d\sigma \right) \\
\leq \hat{C}_0 B_0^{plN} \exp (-\min \left\{ r_f, q \right\} l |y - z_0|) \\
\leq \hat{C}_0 B_0^{plN} \exp \left( -\min \left\{ r_f, q, \frac{p}{2} \right\} (1 - \delta_0) l \right) \text{ for } l \geq 1. 
\tag{3.21}
\]

Since

\[
1 + \delta_0 = 1 + \min \left\{ r_f, q, \frac{p}{2} \right\} \frac{1}{2 (\min \left\{ r_f, q, \frac{p}{2} \right\} + 1)} \\
< \min \left\{ r_f, q, \frac{p}{2} \right\} \left( 1 - \frac{\min \left\{ r_f, q, \frac{p}{2} \right\} - 1}{2 (\min \left\{ r_f, q, \frac{p}{2} \right\} + 1)} \right) \\
= \min \left\{ r_f, q, \frac{p}{2} \right\} (1 - \delta_0),
\]
we may take \(0 < \varepsilon << 1\) such that

\[
(1 + \varepsilon) (1 + \delta_0) < \min \left\{ r_f, q, \frac{p}{2} \right\} (1 - \delta_0).
\]
Then, by (3.18) – (3.21), there exists \(l_1 \geq \max \left\{ \tilde{l}, 1 \right\} \) such that (3.16) holds. Therefore, by (3.12) and (3.14) – (3.16), we can conclude that for any \(l > l_1\),

\[
\sup_{t \geq 0} J_0(t [sw_{y,l} + (1-s) w_{z,y}]) < 2\alpha^\infty \text{ for all } 0 \leq s \leq 1 \text{ and for all } y \in S.
\]

Moreover, by Lemma 2.4, there is a unique \(t_0 (sw_{y,l} + (1-s) w_{z,y}) > 0\) such that

\[
t_0 (sw_{y,l} + (1-s) w_{z,y}) [sw_{y,l} + (1-s) w_{z,y}] \in N_0.
\]

This completes the proof.

**Theorem 3.3.** We have

\[
\alpha_0 = \inf_{u \in N_0} J_0 (u) = \inf_{u \in N^\infty} J^\infty (u) = \alpha^\infty.
\]

Furthermore, equation \((E_0)\) does not admit any solution \(u_0\) such that \(J_0 (u_0) = \alpha_0\).

**Proof.** By Lemma 2.4, there is a unique \(t_0 (sw_{y,l}) > 0\) such that \(t_0 (sw_{y,l}) w_{y,l} \in N_0\) for all \(y \in S\), that is

\[
\|t_0 (sw_{y,l}) w_{y,l}\|_{H^1}^2 + \int_{\partial R^N_+} f |t_0 (sw_{y,l}) w_{y,l}|^q d\sigma = \int_{R^N_+} |t_0 (sw_{y,l}) w_{y,l}|^p d x.
\]

Since

\[
\int_{\partial R^N_+} f |w_{y,l}|^q d\sigma \to 0 \text{ as } l \to \infty,
\]

and

\[
\|w_{y,l}\|_{H^1}^2 = \int_{R^N_+} |w_{y,l}|^p d x = \frac{2p}{p-2} \alpha^\infty \text{ for all } l \geq 0 \text{ and for all } y \in S,
\]

we have \(t_0 (sw_{y,l}) \to 1 \text{ as } l \to \infty\). Thus,

\[
\lim_{l \to \infty} J_0 (t_0 (sw_{y,l}) w_{y,l}) = \alpha^\infty \text{ for all } y \in S,
\]

which implies that

\[
\alpha_0 \leq \inf_{u \in N^\infty} J^\infty (u) = \alpha^\infty.
\]

Let \(u \in N_0\). Then, by Lemma 2.4, \(J_0 (u) = \sup_{t \geq 0} J_0 (tu)\). Moreover, there is a unique \(t^\infty > 0\) such that \(t^\infty u \in N^\infty\). Thus,

\[
J_0 (u) \geq J_0 (t^\infty u) \geq J^\infty (t^\infty u) \geq \alpha^\infty
\]

and so \(\alpha_0 \geq \alpha^\infty\). Therefore,

\[
\alpha_0 = \inf_{u \in N_0} J_0 (u) = \inf_{u \in N^\infty} J^\infty (u) = \alpha^\infty.
\]

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Next, we will show that equation (\(E_0\)) does not admit any solution \(u_0\) such that \(J_0(u_0) = \alpha_0\). Suppose the contrary. Then we can assume that there exists \(u_0 \in N_0\) such that \(J_0(u_0) = \alpha_0\). Then, by Lemma 2.4 (i), \(J_0(u_0) = \sup_{t \geq 0} J_0(tu_0)\). Moreover, there is a unique \(t^\infty(u_0) > 0\) such that \(t^\infty(u_0)u_0 \in N^\infty\). Thus,

\[
\alpha^\infty = \inf_{u \in N_0} J_0(u) = J_0(u_0) \geq J_0(t^\infty(u_0)u_0)
\]

\[
\geq \alpha^\infty + \frac{\int_{\partial R^N_+} f |u_0|^q d\sigma}{q},
\]

which implies that \(\int_{\partial R^N_+} f |u_0|^q d\sigma = 0\) and so \(u_0 \equiv 0\) in \(\{ x \in \partial R^N_+ \mid f(x) \neq 0 \}\), form the condition (\(D1\)). Therefore,

\[
\alpha^\infty = \inf_{u \in N^\infty} J^\infty(u) = J^\infty(tu_0u_0).
\]

Since \(|t^\infty(u_0)u_0| \in N^\infty\) and \(J^\infty(|t^\infty(u_0)u_0|) = J^\infty(t^\infty(u_0)u_0) = \alpha^\infty\), by Willem [39, Theorem 4.3] and the maximum principle, we can assume that \(tu_0u_0\) is a positive solution of \((E^\infty)\), this contradicts

\[
u_0 \equiv 0 \text{ in } \{ x \in \partial R^N_+ \mid f(x) \neq 0 \}.
\]

This completes the proof.

By Theorem 3.3, equation (\(E_0\)) does not admit any solution \(u_0\) such that \(J_0(u_0) = \alpha_0\) and

\[
\alpha_0 = \inf_{u \in N_0} J_0(u) = \inf_{u \in N^\infty} J^\infty(u) = \alpha^\infty.
\]

Furthermore, we have the following result.

**Lemma 3.4.** Suppose that \(\{u_n\}\) is a minimizing sequence for \(J_0\) in \(N_0\). Then

\[
\int_{\partial R^N_+} f |u_n|^q d\sigma = o(1).
\]

Furthermore, \(\{u_n\}\) is a \((PS)_{\alpha^\infty}\)-sequence for \(J^\infty\) in \(H^1(R^N_+)\).

**Proof.** For each \(n\), there is a unique \(t_n > 0\) such that \(t_nu_n \in N^\infty\), that is

\[
t^2_n \|u_n\|_{H^1} = t^p_n \int_{R^N_+} |u_n|^p dx.
\]

Then, by Lemma 2.4 (i),

\[
J_0(u_n) \geq J_0(t_nu_n) = J^\infty(t_nu_n) + \frac{t^q_n}{q} \int_{\partial R^N_+} f |u_n|^q d\sigma
\]

\[
\geq \alpha^\infty + \frac{t^q_n}{q} \int_{\partial R^N_+} f |u_n|^q d\sigma.
\]
Since $J_0(u_n) = \alpha^\infty + o(1)$ from Theorem 3.3, we have
\[
\frac{t_n^q}{q} \int_{\partial \mathbb{R}^N_+} f |u_n|^q d\sigma = o(1).
\]

We will show that there exists $c_0 > 0$ such that $t_n > c_0$ for all $n$. Suppose the contrary. Then we may assume $t_n \to 0$ as $n \to \infty$. Since $J_0(u_n) = \alpha^\infty + o(1)$, by Lemma 2.2, we have $\|u_n\|$ is uniformly bounded and so $\|t_n u_n\|_{H^1} \to 0$ or $J^\infty(t_n u_n) \to 0$, and this contradicts $J^\infty(t_n u_n) \geq \alpha^\infty > 0$. Thus,
\[
\int_{\partial \mathbb{R}^N_+} f |u_n|^q d\sigma = o(1),
\]
which implies that
\[
\|u_n\|_{H^1}^2 = \int_{\mathbb{R}^N_+} |u_n|^p dx + o(1)
\]
and
\[
J^\infty(u_n) = \alpha^\infty + o(1).
\]
Moreover, by Wang and Wu [38, Lemma 7], we have $\{u_n\}$ is a (PS)$_{\alpha^\infty}$–sequence for $J^\infty$ in $H^1(\mathbb{R}^N_+)$. 

For $u \in H^1(\mathbb{R}^N_+)$, we define the center mass function from $N_0$ to the unit ball of $\mathbb{R}^{N-1}$
\[
m(u) = \frac{1}{\|u\|_{L^p(\mathbb{R}^N_+)}^p} \int_{\mathbb{R}^N_+} \frac{x'}{|x'|^p} |u(x', x_N)|^p dx'dx_N.
\]
Clearly, $m$ is continuous from $N_0$ to $B^{N-1}(0,1)$ and $|m(u)| < 1$. Let
\[
\theta_0 = \inf \{ J_0(u) \mid u \in N_0, \ u \geq 0, \ m(u) = 0 \}.
\]
Then we have the following result.

**Lemma 3.5.** There exists $\xi_0 > 0$ such that $\alpha^\infty < \xi_0 \leq \theta_0$.

**Proof.** Suppose the contrary. Then there exists a sequence $\{u_n\} \subset N_0$ and $m(u_n) = 0$ for each $n$, such that $J_0(u) = \alpha^\infty + o(1)$. By Lemma 3.4, we have $\{u_n\}$ is a (PS)$_{\alpha^\infty}$–sequence in $H^1(\mathbb{R}^N_+)$ for $J^\infty$. By the concentration–compactness principle (see Lions [30, 31] or del Pino and Flores [22, proof of proposition 2.1]) and the fact that $\alpha^\infty = \tilde{\alpha}^\infty/2 > 0$, there exist a subsequence $\{u_n\}$, a sequence $\{(x'_n, 0)\} \subset \partial \mathbb{R}^N_+$, and a positive solution $w \in H^1(\mathbb{R}^N_+)$ of equation $(E^\infty)$ such that
\[
\|u_n(x) - w(x - (x'_n, 0))\|_{H^1} \to 0 \text{ as } n \to \infty.
\]
Now we will show that \(|(x_n',0)| \to \infty\) as \(n \to \infty\). Suppose the contrary. Then we may assume that \(\{(x_n',0)\}\) is bounded and \((x_n',0) \to (x_0',0)\) for some \((x_0',0) \in \partial \mathbb{R}^+_N\). Thus, by (3.22),

\[
\int_{\partial \mathbb{R}^+_N} f |u_n|^q \, d\sigma = \int_{\partial \mathbb{R}^+_N} f(x) |w(x - (x_n',0))|^q \, d\sigma + o(1)
\]

\[
= \int_{\partial \mathbb{R}^+_N} f(x + (x_0',0)) |w(x)|^q \, d\sigma + o(1),
\]

this contradicts the result of Lemma 3.4: \(\int_{\partial \mathbb{R}^+_N} f |u_n|^q \, d\sigma = o(1)\). Hence we may assume that \(\frac{x_i'}{|x_i'|} \to e\) as \(n \to \infty\), where \(e \in \mathbb{S}\). Then, by (3.22) and the Lebesgue dominated convergence theorem, we have

\[
0 = \left(\text{Lemma 2.4 and Proposition 3.2, for each } y \in \mathbb{S} \text{ and } l > l_1 \text{ there exists } t_0 (w_{y,l}) > 0 \text{ such that } t_0 (w_{y,l}) w_{y,l} \in N_0\right) \text{, we have }
\]

\[
0 = \frac{m(u_n)}{\|u_n\|^p_{L^p(\mathbb{R}^+_N)}} \int_{\mathbb{R}^+_N} x' |u_n(x',x_N)|^p \, dx \, dx_N
\]

\[
= \frac{\|w\|^p_{L^p(\mathbb{R}^+_N)}}{\|w\|^p_{L^p(\mathbb{R}^+_N)}} \int_{\mathbb{R}^+_N} x' + x_n' |w(x',x_N)|^p \, dx \, dx_N + o(1)
\]

\[
= e + o(1) \text{ as } n \to \infty,
\]

which is a contradiction. Therefore, there exists \(\xi_0 > 0\) such that \(\alpha^\infty < \xi_0 \leq \theta_0\).

By Lemma 2.4 and Proposition 3.2, for each \(y \in \mathbb{S}\) and \(l > l_1\) there exists \(t_0 (w_{y,l}) > 0\) such that \(t_0 (w_{y,l}) w_{y,l} \in N_0\). Moreover, we have the following result.

**Lemma 3.6.** There exists \(l_0 \geq l_1\) such that for any \(l \geq l_0\),

(i) \(\alpha^\infty < J_0(t_0 (w_{y,l}) w_{y,l}) < \xi_0\) for all \(y \in \mathbb{S}\);

(ii) \(\langle m(t_0 (w_{y,l}) w_{y,l}), y \rangle > 0\), for all \(y \in \mathbb{S}\).

**Proof.** (i) Follow from (4.4) – (4.6) and Theorem 3.3.

(ii) For \(x' \in \mathbb{R}^{N-1}\) with \(x' + ly \neq 0\), we have

\[
\frac{x' + ly}{|x' + ly|}, l y = |x' + ly| - \frac{1}{|x' + ly|}(x' + ly, x')
\]

\[
\geq |x' + ly| - |x'| \geq l |y| - 2 |x'| = l - 2 |x'|
\]

Then

\[
\langle m(t_0 (w_{y,l}) w_{y,l}), y \rangle = \frac{1}{l \|w_{y,l}\|^p_{L^p(\mathbb{R}^+_N)}} \int_{\mathbb{R}^+_N} \left(\frac{x'}{|x'|}, l y\right) |w_{y,l}|^p \, dx \, dx_N
\]

\[
= \frac{1}{l \|w\|^p_{L^p(\mathbb{R}^+_N)}} \int_{\mathbb{R}^+_N} \left(\frac{x' + ly}{|x' + ly|}, l y\right) |w|^p \, dx \, dx_N
\]

\[
\geq \frac{1}{l \|w\|^p_{L^p(\mathbb{R}^+_N)}} (l \int_{\mathbb{R}^+_N} |w|^p \, dx \, dx_N - 2 \int_{\mathbb{R}^+_N} |x'| |w|^p \, dx \, dx_N)
\]

\[
= 1 - \frac{2c_0}{l},
\]

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where \( c_0 = \|w\|_{L_p(R^N)}^p \int_{R^N} \|x^p dx dx_N. \) Thus, there exists \( l_0 \geq l_1 \) such that
\[
\langle m(t_0 (w_{y,l}) w_{y,l}), y \rangle \geq 1 - \frac{2c_0}{l} > 0 \text{ for all } l \geq l_0.
\]
This completes the proof.

In the following, we will use Bahri-Li’s minimax argument [9]. Let
\[
\mathcal{B} = \{u \in H^1(R^N) \setminus \{0\} \mid u \geq 0 \text{ and } \|u\|_{H^1} = 1\}.
\]
We define
\[
I_0(u) = \sup_{t \geq 0} J_0(tu) : \mathcal{B} \to \mathbb{R}.
\]
Then, by Lemma 2.4 (iii), for each \( u \in H^1(R^N) \setminus \{0\} \) there exists
\[
t_0(u) = \frac{1}{\|u\|_{H^1}} t_0 \left( \frac{u}{\|u\|_{H^1}} \right) > 0
\]
such that \( t_0(u) u \in \mathcal{N}_0 \) and
\[
I_0(u) = J_0(t_0(u) u) = J_0 \left( t_0 \left( \frac{u}{\|u\|_{H^1}} \right) \frac{u}{\|u\|_{H^1}} \right)
\]
(3.23)
Next, we define a map \( h_0 \) from \( S \) to \( \mathcal{B} \) by
\[
h_0(y) = \frac{w(x - l(y,0))}{\|w(x - l(y,0))\|_{H^1}} = \frac{w_{y,l}}{\|t_0(w_{y,l}) w_{y,l}\|_{H^1}},
\]
where \( y \in S \). Then, by (3.10) and (3.23), for \( l > l_0 \) sufficiently large, we have
\[
I_0(h_0(y)) = J_0(t_0(w_{y,l}) w_{y,l}) < \theta_0 \text{ for all } y \in S.
\]
We define another map \( h^* \) from \( \overline{B^{N-1}(0,1)} \) to \( \mathcal{N}_0 \) by
\[
h^*(sy + (1 - s) z_0) = \frac{sw_{y,l} + (1 - s) w_{z_0,l}}{\|sw_{y,l} + (1 - s) w_{z_0,l}\|_{H^1}}
\]
where \( 0 \leq s \leq 1 \) and \( y \in S \). It is clear that \( h^*|S = h_0 \). It follows from Proposition 3.2 and (3.23) that
\[
I_0(h_0(y)) = J_0(t_0(sw_{y,l} + (1 - s) w_{z_0,l}) [sw_{y,l} + (1 - s) w_{z_0,l}]) < 2\alpha^\infty
\]
(3.24)
for all \( y \in S \). We next define a min-max value. Let
\[
\beta_0 = \inf_{\gamma \in \Gamma} \max_{z \in \overline{B^{N-1}(0,1)}} I_0(\gamma(z)),
\]
(3.25)
where
\[
\Gamma = \left\{ \gamma \in C \left( \overline{B^{N-1}(0,1)}, \mathcal{B} \right) \mid \gamma|S = h_0 \right\}.
\]
(3.26)
Note that \( S = \partial B^{N-1}(0,1) \). Then we have the following result.
Lemma 3.7. We have
\[ \alpha^\infty < \xi_0 \leq \theta_0 \leq \beta_0 < 2\alpha^\infty. \]

**Proof.** By Lemmas 3.5, 3.6, (3.24) and (3.23), we only need to show \( \theta_0 \leq \beta_0 \). For any \( \gamma \in \Gamma \), there exists \( t_0 (\gamma(z)) > 0 \) such that \( t_0 (\gamma(z)) \gamma(z) \in \mathbf{N}_0 \) and
\[ t_0 (\gamma(z)) \gamma(z) = t_0 (w_{z,l}) w_{z,l} \text{ for all } z \in \mathcal{S}. \]
Consider the homotopy \( H(s, z) : [0, 1] \times B^{N-1} (0, 1) \to \mathbb{R} \) defined by
\[ H(s, z) = (1 - s)m(t_0 (\gamma(z)) \gamma(z)) + sI(z), \]
where \( I \) denotes the identity map. Note that \( m(t_0 (\gamma(z)) \gamma(z)) = m(t_0 (w_{z,l}) w_{z,l}) \) for all \( z \in \mathcal{S} \). By Lemma 3.6 (ii), \( H(s, z) \neq 0 \) for \( z \in \mathcal{S} \) and \( s \in [0, 1] \). Therefore,
\[ \deg(m(t_0 (\gamma)), B^{N-1} (0, 1), 0) = \deg(I, B^{N-1} (0, 1), 0) = 1. \]
There exists \( z_0 \in B^{N-1} (0, 1) \) such that
\[ m(t_0 (\gamma(z_0)) \gamma(z_0)) = 0. \]
Hence, for each \( \gamma \in \Gamma_0 \), we have
\[ \theta_0 = \inf \{ J_0(u) | u \in \mathbf{N}_0, u \geq 0, m(u) = 0 \} \leq I_0(\gamma(z_0)) \leq \max_{z \in B^{N-1}(0,1)} I_0(\gamma(z)). \]
This shows that \( \theta_0 \leq \beta_0 \).

Now, we are going to assert that the equation \( (E_0) \) has a positive solution.

**Theorem 3.8.** Equation \( (E_0) \) has a positive solution \( \tilde{u}_0 \) such that \( J_0 (\tilde{u}_0) = \beta_0 \).

**Proof.** By Lemma 3.7 and the minimax principle (see Ambrosetti and Rabinowitz [3]), there exists a sequence \( \{u_n\} \subset \mathcal{B} \) such that
\[
\left\{
\begin{array}{l}
I_0 (u_n) = \beta_0 + o(1), \\
\|I'_0 (u_n)\|_{T_{u_n}} \equiv \sup \{I'_0 (u_n)\phi | \phi \in T_{u_n} \mathcal{B}, \|\phi\|_{H^1} = 1\} = o(1) \text{ as } n \to \infty,
\end{array}\right.
\]
where \( \alpha^\infty < \beta_0 < 2\alpha^\infty \) and \( T_{u_n} \mathcal{B} = \{\phi \in H^1 (\mathbb{R}^N_+) \mid \langle \phi, u_n \rangle = 0\} \). By an argument similar to the proof of proposition 1.7 in Adachi and Tanaka [4], there exists \( t_0 (u_n) > 0 \) such that \( t_0 (u_n) u_n \in \mathbf{N}_0 \) and
\[
\left\{
\begin{array}{l}
J_0 (t_0 (u_n) u_n) = \beta_0 + o(1), \\
J'_0 (t_0 (u_n) u_n) = o(1) \text{ in } H^{-1} (\mathbb{R}^N_+) \text{ as } n \to \infty.
\end{array}\right.
\]
Thus, by Corollary 2.6, Theorem 3.3 and the maximum principle, we can conclude that the equation \( (E_0) \) has a positive solution \( \tilde{u}_0 \) such that \( J_0 (\tilde{u}_0) = \beta_0 \).
4. Existence of positive solutions for $\lambda > 0$

By Lemma 2.4 for each $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ and $\lambda > 0$ there is a unique $t_\lambda(u) \geq \hat{t}_\lambda(u)$ such that $t_\lambda(u)u \in N_\lambda$. Let $w_{y,l}$ be as in (3.3). Then we have the following results.

**Proposition 4.1.** For each $\lambda > 0$, there exists $\hat{t}_1 = \hat{t}_1(\lambda) > 0$ such that for any $l \geq \hat{t}_1$,

$$\sup_{t \geq 0} J_\lambda(tw_{y,l}) < \alpha^\infty \text{ for all } y \in S.$$ 

Furthermore, there is a unique $t_\lambda(w_{y,l}) > 0$ such that $t_\lambda(w_{y,l})w_{y,l} \in N_\lambda$.

**Proof.** We have

$$J_\lambda(tw_{y,l}) = \frac{t^2}{2} \|w_{y,l}\|^2_{H^1} + \frac{t^q}{q} \int_{\partial R^N_+} f |w_{y,l}|^q d\sigma - \frac{t^p}{p} \int_{\mathbb{R}^N_+} g_\lambda |w_{y,l}|^p dx$$

$$= \frac{t^2}{2} \|w\|^2_{H^1} - \frac{t^p}{p} \int_{\mathbb{R}^N_+} w^p dx + \frac{t^q}{q} \int_{\partial R^N_+} f w_{y,l}^q d\sigma - \frac{\lambda t^p}{p} \int_{\mathbb{R}^N_+} a w_{y,l}^p dx \quad (4.1)$$

for all $\lambda > 0$. This implies that $J_\lambda(tw_{y,l}) \to -\infty$ as $t \to \infty$ for all $y \in S$. Thus, there exists $t_1 > 0$ such that for any $l \geq 0$,

$$J_\lambda(tw_{y,l}) < \alpha^\infty \text{ for all } t \geq t_1 \text{ and for all } y \in S. \quad (4.2)$$

Moreover, $J_\lambda(0) = 0 < \alpha^\infty, J_\lambda \in C^2(H^1(\mathbb{R}^N), \mathbb{R})$ and $\|w_{y,l}\|^2_{H^1} = \frac{2p}{p-2} \alpha^\infty$ for all $l \geq 0$, this implies that there exists $t_2 > 0$ such that for any $l \geq 0$,

$$J_\lambda(tw_{y,l}) < \alpha^\infty \text{ for all } 0 \leq t \leq t_2 \text{ and for all } y \in S. \quad (4.3)$$

Moreover, by Brown and Zhang [15] and Willem [39], we know that

$$J^\infty(tw) = \frac{t^2}{2} \|w\|^2_{H^1} - \frac{t^p}{p} \int_{\mathbb{R}^N_+} w^p dx \leq \alpha^\infty \text{ for all } t > 0. \quad (4.4)$$

Thus, by (4.1),

$$J_\lambda(tw_{y,l}) \leq \alpha^\infty + \frac{t^q}{q} \int_{\partial R^N_+} f w_{y,l}^q d\sigma - \frac{\lambda t^p}{p} \int_{\mathbb{R}^N_+} a w_{y,l}^p dx \text{ for all } t > 0. \quad (4.5)$$

By (4.2) and (4.3), we only need to show that there exists $\hat{t}_1 > 0$ such that for any $l > \hat{t}_1$,

$$\sup_{t_2 \leq t \leq t_1} J_\lambda(tw_{y,l}) < \alpha^\infty \text{ for all } y \in S.$$
We set
\[ C_0 = \min_{x \in B_+^N (e_N, \frac{1}{2})} w^p (x) > 0, \]
where \( B_+^N (e_N, \frac{1}{2}) = \{ x \in \mathbb{R}^N_+ \mid |x - e_N| < \frac{1}{2} \} \) and \( e_N = (0, \ldots, 0, 1) \in \mathbb{R}^N \). Then, by the conditions \((D1)\),
\[
\int_{\mathbb{R}^N} a w^p_{y,t} dx = \int_{\mathbb{R}^N} a (x + l (y,0)) w^p (x) dx \geq C_0 \int_{B_+^N (e_N, \frac{1}{2})} a (x + l (y,0)) dx \\
\geq C_0 \exp (-ra l).
\]
Moreover, by \((3.1)\) and the condition \((D1)\),
\[
\int_{\partial \mathbb{R}^N_+} f w^q_{y,t} d\sigma \leq \tilde{c} B_0^p \int_{\partial \mathbb{R}^N_+} \exp (-rf |x|) \exp (-q |x - l (y,0)|) d\sigma \\
\leq C_0 \exp (- \min \{ rf, q \} l) \tag{4.6}
\]
Since \( r_a < \min \{ rf, q \} \) and \( t_2 \leq t \leq t_1 \), we can find \( \tilde{l}_1 > 0 \) such that for any \( l > \tilde{l}_1 \),
\[
\frac{t^q}{q} \int_{\partial \mathbb{R}^N_+} f w^q_{y,t} d\sigma < \frac{\lambda t^p}{p} \int_{\mathbb{R}^N} a w^p_{y,t} dx \text{ for all } y \in S \text{ and for all } t \in [t_2, t_1]. \tag{4.7}
\]
Thus, by \((4.2) - (4.5)\) and \((4.7)\), we obtain that for any \( l > \tilde{l}_1 \)
\[
\sup_{t \geq 0} J_{\lambda} (tw_{y,l}) < \alpha^\infty \text{ for all } y \in S.
\]
Moreover, by Lemma 2.4, there is a unique \( t_\lambda (w_{y,l}) > 0 \) such that \( t_\lambda (w_{y,l}) w_{y,l} \in N_\lambda \).
This completes the proof.

**Theorem 4.2.** For each \( \lambda > 0 \), equation \((E_\lambda)\) has a positive solution \( u_0 \) such that
\[
J_{\lambda} (u_0) = \alpha_\lambda = \inf_{u \in N_\lambda} J_{\lambda} (u) < \alpha^\infty.
\]

**Proof.** By analogy with the proof of Ni and Takagi [32], one can show that the Ekeland variational principle (see [23]), there exists a minimizing sequence \( \{ u_n \} \subset N_\lambda \) such that
\[
J_{\lambda} (u_n) = \alpha_\lambda + o(1) \text{ and } J_{\lambda}' (u_n) = o(1) \text{ in } H^{-1} (\mathbb{R}^N_+) \text{.}
\]
Since \( \inf_{u \in N_\lambda} J_{\lambda} (u) < \alpha^\infty \) from Proposition 4.1 \((ii)\), by Lemma 2.2 and Corollary 2.6 there exist a subsequence \( \{ u_n \} \) and \( u_0 \in N_\lambda \) is a nonzero solution of equation \((E_\lambda)\) such that
\[
u_n \rightarrow u_0 \text{ strongly in } H^1 (\mathbb{R}^N_+) \text{ and } J_{\lambda} (u_0) = \alpha_\lambda.
\]
Since \( J_{\lambda} (u_0) = J_{\lambda} (|u_0|) \) and \( |u_0| \in N_\lambda \), by Lemma 2.3, we may assume that \( u_0 \) is a positive solution of equation \((E_\lambda)\). This completes the proof.
5. Existence of two positive solutions

We need the following result.

**Lemma 5.1.** There exists $d_0 > 0$ such that if $u \in N_0$ and $J_0 (u) \leq \alpha^\infty + d_0$, then

$$\int_{\mathbb{R}^N_+} \frac{x'}{|x'|} \left( (\nabla u)^2 + u^2 \right) dx' dx_N \neq 0.$$

**Proof.** Suppose the contrary. Then there exists sequence $\{u_n\} \subset N_0$ such that $J_0 (u) = \alpha^\infty + o (1)$ and

$$\int_{\mathbb{R}^N_+} \frac{x'}{|x'|} \left( (\nabla u_n)^2 + u_n^2 \right) dx' dx_N = 0.$$

Moreover, by Lemma 3.4, we have $\{u_n\}$ is a $(PS)_{\alpha^\infty}$–sequence in $H^1 (\mathbb{R}^N_+)$ for $J^\infty$. By the concentration–compactness principle (see Lions [30, 31] or del Pino and Flores [22, proof of proposition 2.1]) and $\alpha^\infty = \tilde{\alpha}^\infty / 2$, there exist a subsequence $\{u_n\}$, a sequence $\{(x'_n, 0)\} \subset \partial \mathbb{R}^N_+$, and a positive solution $w \in H^1 (\mathbb{R}^N_+)$ of equation $(E^\infty)$ such that

$$\|u_n (x) - w (x - (x'_n, 0))\|_{H^1} \to 0 \text{ as } n \to \infty. \quad (5.1)$$

Now we will show that $|(x'_n, 0)| \to \infty$ as $n \to \infty$. Suppose the contrary. Then we may assume that $|(x'_n, 0)|$ is bounded and $(x'_n, 0) \to (x'_0, 0)$ for some $(x'_0, 0) \in \partial \mathbb{R}^N_+$. Thus, by (5.1),

$$\int_{\partial \mathbb{R}^N_+} f |u_n|^q d\sigma = \int_{\partial \mathbb{R}^N_+} f (x) |w (x - (x'_n, 0))|^q d\sigma + o (1) = \int_{\partial \mathbb{R}^N_+} f (x + (x'_0, 0)) |w (x)|^q d\sigma + o (1),$$

which contradicts the result of Lemma 3.4: $\int_{\partial \mathbb{R}^N_+} f |u_n|^q d\sigma = o (1)$. Hence we may assume $\frac{x'_n}{|x'_n|} \to e$ as $n \to \infty$, where $e \in S = \{ x' \in \mathbb{R}^{N-1} \mid |x'| = 1 \}$. Then, by the Lebesgue dominated convergence theorem, we have

$$0 = \int_{\mathbb{R}^N_+} \frac{x'}{|x'|} \left( (\nabla u_n)^2 + u_n^2 \right) dx' dx_N = \int_{\mathbb{R}^N_+} \frac{x'}{|x'|} \left( (\nabla u)^2 + u^2 \right) dx + o (1)$$

$$= \frac{2p}{p - 2} \alpha^\infty e + o (1),$$

which is a contradiction. This completes the proof.

For $u \in N_\lambda$, by Lemma 2.4, there is a unique $t_0 (u) > 0$ such that $t_0 (u) u \in N_0$. Let $\eta_0 = (1 + \lambda \|a\|_{\infty})^{\frac{1}{p - 2}}$. Then we have the following result.

**Lemma 5.2.** We have $t_0 (u) < \eta_0$ for all $u \in N_\lambda$. 

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Proof. Let \( u \in N_\lambda \). Then we have
\[
\|u\|_{H^1}^2 + \int_{\partial \mathbb{R}^N_+} f |u|^q \, d\sigma = \int_{\mathbb{R}^N_+} g_\lambda |u|^p \, dx.
\]
We distinguish two cases.

Case (A) : \( t_0(u) < 1 \). Since \( \theta_0 > 1 \), we have
\[
t_0(u) < 1 < \eta_0.
\]

Case (B) : \( t_0(u) \geq 1 \). Since
\[
[t_0(u)]^p \int_{\mathbb{R}^N_+} |u|^p \, dx = [t_0(u)]^2 \|u\|_{H^1}^2 + [t(u)]^q \int_{\partial \mathbb{R}^N_+} f |u|^q \, d\sigma
\]
\[
\leq [t_0(u)]^q \left( \|u\|_{H^1}^2 + \int_{\partial \mathbb{R}^N_+} f |u|^q \, d\sigma \right)
\]
\[
= [t_0(u)]^q \int_{\mathbb{R}^N_+} g_\lambda |u|^p \, dx
\]
\[
\leq [t_0(u)]^q (1 + \lambda \|a\|_\infty) \int_{\mathbb{R}^N_+} |u|^p \, dx,
\]
we have
\[
t_0(u) \leq (1 + \lambda \|a\|_\infty)^{\frac{1}{p-q}} = \eta_0.
\]
This completes the proof.

By the proof of Proposition 4.1, there exist positive numbers \( t_\lambda (w_{y,l}) \) and \( \hat{l}_1 \) such that \( t_\lambda (w_{y,l}) \in N_\lambda \) and
\[
J_\lambda(t_\lambda (w_{y,l}) w_{y,l}) < \alpha^\infty \text{ for all } l > \hat{l}_1.
\]
Then we have the following result.

**Lemma 5.3.** There exists a positive number \( \lambda_0 \) such that for every \( \lambda \in (0, \lambda_0) \), we have
\[
\int_{\mathbb{R}^N_+} \frac{x'}{|x'|} (|\nabla u|^2 + u^2) \, dx' \, dx_N \neq 0
\]
for all \( u \in N_\lambda \) with \( J_\lambda(u) < \alpha^\infty \).

Proof. (i) Let \( u \in N_\lambda \) with \( J_\lambda(u) < \alpha^\infty \). Then, by Lemma 2.4 (i), there exists \( t_0(u) > 0 \) such that \( t_0(u) u \in N_0 \). Moreover,

\[
J_\lambda(u) = \sup_{t \geq 0} J_\lambda(tu) \geq J_\lambda(t_0(u)u)
\]
\[
= J_0(t_0(u)u) - \lambda [t_0(u)]^p \int_{\mathbb{R}^N_+} a |u|^p \, dx.
\]
Thus, by Lemma 5.2 and the Sobolev inequality,

\[ J_0 (t_0 (u) u) \leq J_\lambda (u) + \lambda |t_0 (u)|^p \int_{\mathbb{R}^N_+} a |u|^p \, dx \]

\[ < \alpha^\infty + \lambda c_0 \eta_0^p \|a\|_\infty \|u\|_{H^1}^p \quad \text{for some} \ c_0 > 0. \quad (5.2) \]

Moreover, by (2.1),

\[ \alpha^\infty > J_\lambda (u) \geq \frac{p - 2}{2p} \|u\|^2_{H^1}, \]

which implies that

\[ \|u\|_{H^1} < \left( \frac{2p\alpha^\infty}{p - 2} \right)^{\frac{1}{2}} \quad (5.3) \]

for all \( u \in N_\lambda \) with \( J_\lambda (u) < \alpha^\infty \). Therefore, by (5.2) and (5.3),

\[ J_0 (t_0 (u) u) < \alpha^\infty + \lambda c_0 \eta_0^p \|a\|_\infty \left( \frac{2p\alpha^\infty}{p - 2} \right)^{\frac{1}{2}}. \]

Let \( d_0 > 0 \) be as in Lemma 5.1. Then there exists a positive number \( \lambda_0 \) such that for \( \lambda \in (0, \lambda_0) \),

\[ J_0 (t (u) u) < \alpha^\infty + d_0. \quad (5.4) \]

Since \( t_0 (u) u \in N_0 \) and \( t_0 (u) > 0 \), by Lemma 5.1 and (5.4)

\[ \int_{\mathbb{R}^N_+} x' \left( |\nabla (t_0 (u) u)|^2 + (t_0 (u) u)^2 \right) \, dx' \, dx_N \neq 0, \]

which implies that

\[ \int_{\mathbb{R}^N_+} \frac{x'}{|x'|} \left( |\nabla u|^2 + u^2 \right) \, dx' \, dx_N \neq 0 \]

for all \( u \in N_\lambda \) with \( J_\lambda (u) < \alpha^\infty \).

In the following, we use an idea of Adachi and Tanaka [4]. For \( c \in \mathbb{R}^+ \), we denote

\[ [J_\lambda \leq c] = \{ u \in N_\lambda \mid u \geq 0, J_\lambda (u) \leq c \}. \]

We then try to show for a sufficiently small \( \sigma > 0 \)

\[ \text{cat} ([J_\lambda \leq \alpha^\infty - \sigma]) \geq 2. \quad (5.5) \]

To prove (5.5), we need some preliminaries. Recall the definition of Lusternik-Schnirelman category.

**Definition 5.4.** (i) For a topological space \( X \), we say a non-empty, closed subset \( Y \subseteq X \) is contractible to a point in \( X \) if and only if there exists a continuous mapping

\[ \xi : [0, 1] \times Y \rightarrow X \]

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such that for some \( x_0 \in X \)

\[
\xi(0,x) = x \text{ for all } x \in Y,
\]

and

\[
\xi(1,x) = x_0 \text{ for all } x \in Y.
\]

(ii) We define

\[
\text{cat}(X) = \min \{ k \in \mathbb{N} \mid \text{there exist closed subsets } Y_1, \ldots, Y_k \subset X \text{ such that } Y_j \text{ is contractible to a point in } X \text{ for all } j \text{ and } \bigcup_{j=1}^{k} Y_j = X \}.
\]

When there do not exist finitely many closed subsets \( Y_1, \ldots, Y_k \subset X \) such that \( Y_j \) is contractible to a point in \( X \) for all \( j \) and \( \bigcup_{j=1}^{k} Y_j = X \), we say \( \text{cat}(X) = \infty \).

We need the following two lemmas.

**Lemma 5.5.** Suppose that \( X \) is a Hilbert manifold and \( F \in C^1(X, \mathbb{R}) \). Assume that there are \( c_0 \in \mathbb{R} \) and \( k \in \mathbb{N} \),

(i) \( F(x) \) satisfies the Palais–Smale condition for energy level \( c \leq c_0 \);

(ii) \( \text{cat}(\{ x \in X \mid F(x) \leq c_0 \}) \geq k. \)

Then \( F(x) \) has at least \( k \) critical points in \( \{ x \in X ; F(x) \leq c_0 \} \).

**Proof.** See Ambrosetti [1, Theorem 2.3].

Let \( S^{m-1} = \{ x \in \mathbb{R}^m \mid |x| = 1 \} \) be a unit sphere in \( \mathbb{R}^m \) for \( m \in \mathbb{N} \). Then we have the following results.

**Lemma 5.6.** Let \( X \) be a topological space. Suppose that there are two continuous maps

\[
\Phi : S^{m-1} \rightarrow X, \quad \Psi : X \rightarrow S^{m-1}
\]

such that \( \Psi \circ \Phi \) is homotopic to the identity map of \( S^{m-1} \), that is, there exists a continuous map \( \zeta : [0,1] \times S^{m-1} \rightarrow S^{m-1} \) such that

\[
\zeta(0,x) = (\Psi \circ \Phi)(x) \text{ for each } x \in S^{m-1},
\]

\[
\zeta(1,x) = x \text{ for each } x \in S^{m-1}.
\]

Then

\[
\text{cat}(X) \geq 2.
\]

**Proof.** See Adachi and Tanaka [4, Lemma 2.5].

For \( l > \hat{l}_1 \), we may define a map \( \Phi_{\lambda,l} : S^{(N-1)-1} \rightarrow H^1(\mathbb{R}^N_+) \) by

\[
\Phi_{\lambda,l}(y)(x) = t_{\lambda}(w(x - l(y,0))) w(x - l(y,0)) \text{ for } y \in S^{(N-1)-1},
\]

where \( t_{\lambda}(w(x - l(y,0))) w(x - l(y,0)) \) is as in the proof of Proposition 4.1. Note that \( S^{(N-1)-1} = S \). Then we have the following result.
Lemma 5.7. There exists a sequence \( \{ \sigma_l \} \subset \mathbb{R}^+ \) with \( \sigma_l \to 0 \) as \( l \to \infty \) such that
\[
\Phi_{\lambda,l} \left( S^{(N-1)-1} \right) \subset [J_\lambda \leq \alpha^\infty - \sigma_l].
\]

Proof. By Proposition 4.1, for each \( l > \hat{l}_1 \) we have
\[
t_\lambda \left( w \left( x - l (y,0) \right) \right) w \left( x - l (y,0) \right) \in N_\lambda
\]
and
\[
\sup_{l > \hat{l}_1} J_\lambda \left( t_\lambda \left( w \left( x - l (y,0) \right) \right) w \left( x - l (y,0) \right) \right) < \alpha^\infty \text{ for all } y \in S^{(N-1)-1}.
\]
Since \( S = S^{(N-1)-1} \) and \( \Phi_{\lambda,l} \left( S^{(N-1)-1} \right) \) is compact,
\[
J_\lambda \left( t_\lambda \left( w \left( x - l (y,0) \right) \right) w \left( x - l (y,0) \right) \right) \leq \alpha^\infty - \sigma_l,
\]
so that the conclusion holds.

From Lemma 5.3, we define
\[
\Psi_\lambda : [J_\lambda < \alpha^\infty] \to S^{(N-1)-1}
\]
by
\[
\Psi_\lambda (u) = \frac{\int_{\mathbb{R}^N_+} \frac{x'}{|x'|} \left( |\nabla u|^2 + u^2 \right) dx'dx_N}{\left| \int_{\mathbb{R}^N_+} \frac{x'}{|x'|} \left( |\nabla u|^2 + u^2 \right) dx'dx_N \right|}.
\]
Then we have the following results.

Lemma 5.8. Let \( \lambda_0 > 0 \) be as in Lemma 5.3. Then for each \( \lambda \in (0, \lambda_0) \) and there exists \( \hat{l}_0 \geq \hat{l}_1 \) such that for \( l > \hat{l}_0 \), the map
\[
\Psi_\lambda \circ \Phi_{\lambda,l} : S^{(N-1)-1} \to S^{(N-1)-1}
\]
is homotopic to the identity.

Proof. Let \( \Sigma = \left\{ u \in H^1 \left( \mathbb{R}^N_+ \right) \setminus \{0\} \mid \int_{\mathbb{R}^N_+} \frac{x'}{|x'|} \left( |\nabla u|^2 + u^2 \right) dx'dx_N \neq 0 \right\} \). We define
\[
\overline{\Psi}_\lambda : \Sigma \to S^{(N-1)-1}
\]
by
\[
\overline{\Psi}_\lambda (u) = \frac{\int_{\mathbb{R}^N_+} \frac{x'}{|x'|} \left( |\nabla u|^2 + u^2 \right) dx'dx_N}{\left| \int_{\mathbb{R}^N_+} \frac{x'}{|x'|} \left( |\nabla u|^2 + u^2 \right) dx'dx_N \right|},
\]
an extension of \( \Psi_\lambda \). Since \( w \left( x - l (y,0) \right) \in \Sigma \) for all \( e \in S^{(N-1)-1} \) and for \( l \) sufficiently large, we let \( \gamma : [s_1, s_2] \to S^{(N-1)-1} \) be a regular geodesic between \( \overline{\Psi}_\lambda (w_{y,l}) \) and

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Thus, \( \gamma (s_1) = \Psi_\lambda (w_{y,l}), \gamma (s_2) = \Psi_\lambda (\Phi_{\lambda,l} (y)) \). By an argument similar to that in Lemma 5.1, there exists a positive number \( \hat{l}_0 \geq \hat{l}_1 \) such that for \( l > \hat{l}_0 \),

\[
\begin{align*}
    w \left( x - \frac{l(y,0)}{2(1-\theta)} \right) \in \Sigma \text{ for all } y \in S^{(N-1)-1} \text{ and } \theta \in [1/2, 1].
\end{align*}
\]

We define \( \zeta_l (\theta, y) : [0, 1] \times S^{(N-1)-1} \rightarrow S^{(N-1)-1} \) by

\[
\begin{align*}
    \zeta_l (\theta, y) = \begin{cases} 
        \gamma (2\theta (s_1 - s_2) + s_2) & \text{for } \theta \in [0, 1/2); \\
        \Psi_\lambda \left( w \left( x - \frac{l(y,0)}{2(1-\theta)} \right) \right) & \text{for } \theta \in [1/2, 1); \\
        y & \text{for } \theta = 1.
    \end{cases}
\end{align*}
\]

Then \( \zeta_l (0, y) = \Psi_\lambda (\Phi_{\lambda,l} (y)) = \Psi_\lambda (\Phi_{\lambda,l} (y)) \) and \( \zeta_l (1, y) = y \). First, we claim that \( \lim_{\theta \rightarrow 1^-} \zeta_l (\theta, y) = y \) and \( \lim_{\theta \rightarrow 1^-} \zeta_l (\theta, y) = \Psi_\lambda (w_{y,l}) \).

(a) \( \lim_{\theta \rightarrow 1^-} \zeta_l (\theta, y) = y \) : since

\[
\begin{align*}
    \int_{\mathbb{R}_N^N} \frac{x'}{|x'|} \left( \left| \nabla \left[ w \left( x - \frac{l(y,0)}{2(1-\theta)} \right) \right] \right|^2 + \left[ w \left( x - \frac{l(y,0)}{2(1-\theta)} \right) \right]^2 \right) dx' dx_N \\
    = \int_{\mathbb{R}_N^N} \frac{x'}{|x'|} \left[ \left| \nabla [w(x)] \right|^2 + [w(x)]^2 \right] dx' dx_N \\
    = \left( \frac{2p}{p-2} \right) \alpha^\infty y + o(1) \text{ as } \theta \rightarrow 1^-,
\end{align*}
\]

then \( \lim_{\theta \rightarrow 1^-} \zeta_l (\theta, y) = y \).

(b) \( \lim_{\theta \rightarrow 1^-} \zeta_l (\theta, y) = \Psi_\lambda (w_{y,l}) \) : since \( \Psi_\lambda \in C \left( \Sigma, S^{(N-1)-1} \right) \), we obtain \( \lim_{\theta \rightarrow 1^-} \zeta_l (\theta, y) = \Psi_\lambda (w_{y,l}) \).

Thus, \( \zeta_l (\theta, y) \in C \left( [0, 1] \times S^{(N-1)-1}, S^{(N-1)-1} \right) \) and

\[
\begin{align*}
    \zeta_l (0, y) &= \Psi_\lambda (\Phi_{\lambda,l} (y)) \text{ for all } y \in S^{(N-1)-1}, \\
    \zeta_l (1, y) &= y \text{ for all } y \in S^{(N-1)-1},
\end{align*}
\]

provided \( l > \hat{l}_0 \). This completes the proof.

**Theorem 5.9.** For each \( \lambda \in (0, \lambda_0) \), functional \( J_\lambda \) has at least two critical points in \([J_\lambda < \alpha^\infty]\). In particular, equation \((E_\lambda)\) has two positive solutions \( u_0^{(1)} \) and \( u_0^{(2)} \) such that \( u_0^{(i)} \in N_\lambda \) for \( i = 1, 2 \).
Proof. Applying Lemmas 5.6 and 5.8, we have for \( \lambda \in (0, \lambda_0) \),

\[
\text{cat} ([J_\lambda \leq \alpha^\infty - \sigma_i]) \geq 2.
\]

By Corollary 2.6 and Lemmas 5.5 and 5.7, \( J_\lambda(u) \) has at least two critical points in \( [J_\lambda < \alpha^\infty] \). This implies, equation \((E_\lambda)\) has two nontrivial nonnegative solutions \( u_1 \) and \( u_2 \) such that \( u_i \in N_\lambda \) for \( i = 1, 2 \). Moreover, by the maximum principle, we have \( u_i > 0 \) in \( \mathbb{R}^N_+ \).

6. Proof of Theorem 1.1

Given a positive real number \( r_0 > \max \left\{ \frac{2}{p-2}, \frac{q}{p-q} \right\} \). Let

\[
\Lambda_0 = \min \left\{ \frac{r_0 (p - 2)}{2 \|a\|_\infty (r_0 + 1)}, \frac{r_0 (p - q)}{q \|a\|_\infty (r_0 + 1)}, \lambda_0 \right\},
\]

where \( \lambda_0 > 0 \) as in Lemma 5.3. Then we have the following results.

Lemma 6.1. We have

\[
\frac{1}{2} (1 + \lambda \|a\|_\infty)^{r_0} - \frac{1}{p} (1 + \lambda \|a\|_\infty)^{r_0 + 1} - \frac{p - 2}{2p} > 0
\]

and

\[
\frac{1}{q} (1 + \lambda \|a\|_\infty)^{r_0} - \frac{1}{p} (1 + \lambda \|a\|_\infty)^{r_0 + 1} - \frac{p - q}{pq} > 0
\]

for all \( \lambda \in (0, \Lambda_0) \).

Proof. Let

\[
k(\lambda) = \frac{1}{2} (1 + \lambda \|a\|_\infty)^{r_0} - \frac{1}{p} (1 + \lambda \|a\|_\infty)^{r_0 + 1} - \frac{p - 2}{2p}.
\]

Then \( k(0) = 0 \) and

\[
k'(\lambda) = \frac{r_0}{2} (1 + \lambda \|a\|_\infty)^{r_0 - 1} \|a\|_\infty - \frac{r_0 + 1}{p} (1 + \lambda \|a\|_\infty)^{r_0} \|a\|_\infty
\]

\[
= \|a\|_\infty (1 + \lambda \|a\|_\infty)^{r_0 - 1} \left( \frac{r_0}{2} - \frac{r_0 + 1}{p} (1 + \lambda \|a\|_\infty) \right) > 0
\]

for all \( \lambda \in (0, \Lambda_0) \). This implies that \( k(\lambda) > 0 \) or

\[
\frac{1}{2} (1 + \lambda \|a\|_\infty)^{r_0} - \frac{1}{p} (1 + \lambda \|a\|_\infty)^{r_0 + 1} - \frac{p - 2}{2p} > 0 \text{ for all } \lambda \in (0, \Lambda_0).
\]

Similar to the argument we also have

\[
\frac{1}{q} (1 + \lambda \|a\|_\infty)^{r_0} - \frac{1}{p} (1 + \lambda \|a\|_\infty)^{r_0 + 1} - \frac{p - q}{pq} > 0 \text{ for all } \lambda \in (0, \Lambda_0).
\]
This completes the proof.

We define

\[ I_\lambda(u) = \sup_{t \geq 0} J_\lambda(tu) \]: \( B \to \mathbb{R} \).

Then we have the following result.

**Lemma 6.2.** For each \( \lambda \in (0, \Lambda_0) \) and \( u \in B \) we have

\[
(1 + \lambda \|a\|_\infty)^{-r_0} I_0(u) \leq I_\lambda(u) \leq I_0(u).
\]

**Proof.** Let \( u \in B \). Then, by Lemmas 2.4 and 6.1, and (3.23)

\[
I_\lambda(u) = \sup_{t \geq 0} J_\lambda(tu) \geq J_\lambda(t_0(u)u)
\]

\[
\geq \frac{1}{2} \int_{\mathbb{R}_+^N} |\nabla t_0(u)u|^2 + (t_0(u)u)^2\,dx + \frac{1}{q} \int_{\partial \mathbb{R}_+^N} f|t_0(u)u|^q\,d\sigma
\]

\[
\geq \left( \frac{1}{2} - \frac{1 + \lambda \|a\|_\infty}{p} \right) \int_{\mathbb{R}_+^N} |\nabla t_0(u)u|^2 + (t_0(u)u)^2\,dx
\]

\[
\geq \left( \frac{1}{2} - \frac{1 + \lambda \|a\|_\infty}{p} \right) \int_{\mathbb{R}_+^N} |\nabla t_0(u)u|^2 + (t_0(u)u)^2\,dx
\]

\[
\geq (1 + \lambda \|a\|_\infty)^{-r_0} J_0(t_0(u)u) = (1 + \lambda \|a\|_\infty)^{-r_0} I_0(u).
\]

Moreover,

\[
J_\lambda(tu) \leq J_0(tu) \leq I_0(u) \text{ for all } t > 0.
\]

Then \( I_\lambda(u) \leq I_0(u) \). This completes the proof.

We observe that if \( \lambda \) is sufficiently small, the minimax argument in Section 4 also works for \( J_\lambda \). Let \( l > \max \{ l_0, \hat{l}_0 \} \) be very large and let

\[
\beta_\lambda = \inf_{\gamma \in \Gamma} \max_{y \in B^{N-1}(0, 1)} I_\lambda(\gamma(y)).
\]

where \( \Gamma \) is as in (3.26). Then, by (3.25) and Lemma 6.2, for \( \lambda \in (0, \Lambda_0) \), we have

\[
(1 + \lambda \|a\|_\infty)^{-r_0} \beta_0 \leq \beta_\lambda \leq \beta_0.
\]

Then we have the following result.

**Theorem 6.3.** There exists a positive number \( \lambda_* \leq \Lambda_0 \) such that for \( \lambda \in (0, \lambda_*) \),

\[
\alpha^\infty < \beta_\lambda < \alpha^\infty + \alpha_\lambda < 2\alpha^\infty.
\]

Furthermore, equation \((E_\lambda)\) has a positive solution \( u_0^{(3)} \) such that \( J_\lambda(u_0^{(3)}) = \beta_\lambda \).
Proof. By Theorems 3.3 and 4.2, and Lemma 6.2, we also have that

\[(1 + \lambda \|a\|_\infty)^{-r_0} \alpha^\infty \leq \alpha_\lambda < \alpha^\infty.\]

For any \(\varepsilon > 0\) there exists a positive number \(\overline{\lambda}_1 \leq \Lambda_0\) such that for \(\lambda \in (0, \overline{\lambda}_1)\),

\[\alpha^\infty - \varepsilon < \alpha_\lambda < \alpha^\infty.\]

Thus,

\[2\alpha^\infty - \varepsilon < \alpha^\infty + \alpha_\lambda < 2\alpha^\infty.\]

Applying (6.1) for any \(\delta > 0\) there exists a positive number \(\overline{\lambda}_2 \leq \Lambda_0\) such that for \(\lambda \in (0, \overline{\lambda}_2)\),

\[\beta_0 - \delta < \beta_\lambda \leq \beta_0.\]

Moreover, by Lemma 3.7,

\[\alpha^\infty < \beta_0 < 2\alpha^\infty.\]

Fix a small \(0 < \varepsilon < 2\alpha^\infty - \beta_0\), choosing a \(\delta > 0\) such that for \(\lambda \in (0, \lambda_*)\) we get,

\[\alpha^\infty < \beta_\lambda < 2\alpha^\infty - \varepsilon < \alpha^\infty + \alpha_\lambda < 2\alpha^\infty,\]

where \(\lambda_* = \min \{\overline{\lambda}_1, \overline{\lambda}_2\}\). Similar to the argument in the proof of Theorem 3.8, we can conclude that the equation \((E_{\lambda})\) has a positive solution \(u_0^{(3)}\) such that \(J_\lambda (u_0^{(3)}) = \beta_\lambda\). This completes the proof.

We can now complete the proof of Theorem 1.1: By Theorems 4.2 and 3.8, equation \((E_{\lambda})\) has at least one positive solution for all \(\lambda \in [0, \infty)\). Moreover, by Theorems 5.9 and 6.3, there exists a positive number \(\lambda_*\) such that for \(\lambda \in (0, \lambda_*)\), equation \((E_{\lambda})\) has three positive solutions \(u_0^{(1)}, u_0^{(2)}\) and \(u_0^{(3)}\) with

\[0 < J_\lambda (u_0^{(i)}) < \alpha^\infty < J_\lambda (u_0^{(3)})\]

for \(i = 1, 2\).

This completes the proof of Theorem 1.1.

References


